

THE CONJUGACY PROBLEM FOR POSITIVE HOMOGENEOUSLY PRESENTED MONOIDS.

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ABSTRACT. Let M be a positive homogeneously presented monoid $\langle L \mid R \rangle_{mo}$. If M satisfies the cancellation condition and carries certain particular elements similar to the *fundamental elements* in Artin monoids, then the solvability of the conjugacy problem for M implies that in the corresponding group $\langle L \mid R \rangle$. In addition to these conditions, if M satisfies the LCM condition (i.e. any two elements α and β in M admit the left (resp. right) least common multiple), then the solution to the conjugacy problem for M is known. We will give two kinds of examples that do not satisfy only the LCM condition. For these examples, we will give a solution to the conjugacy problem by improving the method given by E. Brieskorn and K. Saito.

1. INTRODUCTION

Let M be a positive homogeneously presented monoid $\langle L \mid R \rangle_{mo}$. In this article, for the monoid M that satisfies the cancellation condition and carries certain particular elements similar to the *fundamental elements* in Artin monoids, we will discuss how to solve the conjugacy problem. In addition to these two conditions, if M satisfies the LCM condition, the solution to the conjugacy problem has already been known ([B-S], [D-P], [P]). We will deal with two kinds of examples $G_{B_{ii}}^+$ and $G_{m,n}^+$ that do not satisfy only the LCM condition. By improving the method in [B-S], we will give a solution to the conjugacy problem for them. As a consequence, the conjugacy problem in the corresponding groups can be solved.

In [B-S], the notion of the *Artin groups* was introduced as a generalization of the notion of the braid groups ([A1], [A2]). In [De], the Artin groups were called by the alias of the *generalized braid groups*. The Artin group appears as the fundamental group of the regular orbit space of a finite reflection group W ([B]). E. Brieskorn gave a special presentation of the fundamental group whose defining relations correspond to the finite Coxeter diagram of type W . The monoid defined by that presentation is called *Artin monoid* ([B-S]). In [B-S], by referring to the method in [G], they showed that the Artin monoid is *cancellative* (i.e. $axb = ayb$ implies $x = y$) and that, for any two elements in the monoid, left (resp. right) common multiples exist. Hence, due to the Öre's criterion (see [C-P]), the Artin monoid injects in the corresponding Artin group. Furthermore, they showed that the Artin monoid satisfies the LCM condition (see [B-S] §4). By using this property, in the monoid, they defined a particular element Δ , the *fundamental element*, as the least common multiple of all the generators. We note that the cancellativity and the existence of the least common multiple are the consequence of the *reduction lemma* (see [B-S] §2). By using these properties, they proved that the solvability of the conjugacy problem in the Artin monoid implies that in the corresponding Artin

group. Through the solution of the conjugacy problem for the Artin monoid, the conjugacy problem for the Artin group was solved.

After this work, in the late 90's, the notion of Artin group (resp. Artin monoid) is generalized by French mathematicians ([D-P], [D1]), which is called the *Garside group* (resp. *Garside monoid*). The Garside group is defined as the group of fractions of a Garside monoid. A Garside monoid is a finitely generated monoid that satisfies the following conditions: i) the monoid is cancellative; ii) *atomic* (i.e. the expressions of a given element have bounded lengths); iii) the LCM condition is satisfied; iv) a *Garside element* exists. Hence, the Garside monoid trivially satisfies the Ore's criterion. In §2, we will generalize the notion of fundamental element and, for simplicity, also call it a *fundamental element*. Namely, we call an element Δ in a positive homogeneously presented monoid $M = \langle L \mid R \rangle_{mo}$ fundamental element, if there exists a permutation σ of the set L/\sim (\sim = the image of the set L in M) such that

$$\Delta = s \cdot \Delta = \Delta \cdot \sigma(s)$$

holds for all generators $s \in L/\sim$.¹ This notion does not always coincide with the least common multiple of all the generators. We note that, under the assumption that the monoid is cancellative, an element Δ in the monoid is a Garside element if and only if the element Δ is a fundamental element (Proposition 2.2). We will show that, for a Garside monoid, there is a unique fundamental element $\hat{\Delta}$ that divides any fundamental elements in the monoid from the left and the right (Proposition 2.5). The set $\mathcal{F}(M)$ of all the fundamental elements of M forms a subsemigroup of M such that $\mathcal{QZ}(M)\mathcal{F}(M) = \mathcal{F}(M)\mathcal{QZ}(M) = \mathcal{F}(M)$, where $\mathcal{QZ}(M)$ is the set of all the quasi-central elements of M .² From Proposition 2.5, we say that the two-sided idealistic subsemigroup $\mathcal{F}(M)(\subseteq \mathcal{QZ}(M))$ is singly generated by $\hat{\Delta}$. Moreover, in Proposition 2.8, we will show that any indecomposable quasi-central element Δ divides $\hat{\Delta}$ from the left and the right (we call this property *tame*).³ For a Garside group, the conjugacy problem can be solved ([P]), by improving the method in [B-S]. In [D-P], by showing a Garside group is a biautomatic group, they showed that the conjugacy problem is solvable. In [P], the author showed that the LCM condition implies the GCD condition (i.e. for arbitrary two elements in the monoid the left (resp. right) greatest common divisors exist). Then, due to the cancellativity and the LCM condition and the GCD condition, for an arbitrary element in a Garside group the author succeeded in constructing a normal form of it. As a consequence, the conjugacy problem in a Garside group is solved. In §2, we will show that, in a positively presented atomic monoid that carries a fundamental element, the LCM condition is satisfied if and only if the GCD condition is satisfied (Proposition 2.4).

Since the LCM condition is a strong assumption, some Zariski-van Kampen monoids do not satisfy the LCM condition ([B-M][I1][I2][S-I]). As far as we know, for non-abelian positive homogeneously presented monoids that do not satisfy the condition iii), there are few examples for which the cancellativity of them has been shown, since the pre-existing technique is not perfect ([G][B-S][D2][D3]). In

¹In §2, we will deal with a positively presented atomic monoid. Here we deal with a positive homogeneously presented monoid for simplicity.

²An element $\Delta \in M$ is called quasi-central ([B-S, 7.1]) if $s \cdot \Delta = \Delta \cdot \sigma(s)$ for $s \in L$.

³A positive homogeneously presented cancellative monoid that carries a fundamental element is called tame, if for any indecomposable quasi-central element Δ_0 in the monoid there exists a minimal fundamental element Δ such that the element Δ_0 divides Δ from the left and the right.

the papers [I1][I2][S-I], we deal with non-abelian positive homogeneously presented monoids that do not satisfy only the condition iii) and succeeded in showing cancellativity of them by improving the method. Besides the papers [I1][I2][S-I], for non-abelian positive homogeneously presented monoids that do not satisfy the condition iii), there are known a few examples for which the cancellativity of them has been shown (e.g. [D4]). We note that the cancellativity of these examples are shown by the pre-existing method (the consequence of the reduction lemma). In [I1], for the monoid, called the type B_{ii} , that does not satisfy only the condition iii), the author has solved the word problem and the conjugacy problem, and determined the center of it by showing the monoid injects in the corresponding group. In [I3], the author constructed other examples $G_{m,n}^+$ that do not satisfy only the condition iii). Due to the cancellativity and the existence of fundamental elements, the word problem for them can be solved and the center of them was determined. As a consequence, the word problem in the corresponding groups can be solved and the center of them was determined. The conjugacy problem for them has not been solved yet since the former solution ([B-S], [D-P], [P]) depends on the existence of the least common multiple. We need to extend the technique to solve the conjugacy problem. We remark that, in [I1], the author gave a solution to the conjugacy problem by using a very special type of normal form in the monoid. It is desirable that we give other solution that does not depend on the special context.

In this article, we will give a solution to the conjugacy problem in the monoids $G_{B_{ii}}^+$ and $G_{m,n}^+$ by improving the method in [B-S]. Let M' be a positive homogeneously presented cancellative monoid that does not satisfy only the condition iii). From the proposition 2.4, the GCD condition for M' is also not satisfied. Since neither the LCM condition nor the GCD condition is satisfied, we explore a solution to the conjugacy problem for M' that depends on the reduction lemma. We remark that the reduction lemma knows the data for not only the cancellativity of the monoid but also the structure of the left (resp. right) minimal common multiples. For an element w in a positive homogeneously presented cancellative monoid that carries a fundamental element, we call A in the monoid a *right transit element* of w , if the element A is not ε and there exists an element Q such that an equation

$$AQ = wA$$

holds. A right transit element A of w is called *minimal*, if any right transit element of w dividing A from the left coincides with A itself. The set of all right minimal transit elements of w shall be denoted by $\text{Trans}^{\min}(w)$. Due to the homogeneity of the defining relations of the monoid, to solve the conjugacy problem for it, it is sufficient to show that, for an arbitrary element w in it, we decide the set $\text{Trans}^{\min}(w)$ explicitly. In §3, due to the reduction lemma for the monoid $G_{B_{ii}}^+$, we will decide the set $\text{Trans}^{\min}(w)$ for an arbitrary element w in it explicitly. In the papers [B-S], [P], the authors showed that for an arbitrary element w in the monoid the set $\text{Trans}^{\min}(w)$ only consists of left divisors of the unique fundamental element $\hat{\Delta}$ in the monoid, which will be called the property $(P(w; \hat{\Delta}))$ with respect to w . In §4, due to the reduction lemma for the monoid $G_{m,n}^+$, we will show that, for an arbitrary element w in the monoid, the property $(P(w; \hat{\Delta}))$ is satisfied. We remark that the idealistic subsemigroup $\mathcal{F}(G_{m,n}^+)(\subseteq \mathcal{QZ}(G_{m,n}^+))$ is singly generated and the monoid $G_{m,n}^+$ is tame. By generalizing this framework, we conjecture that,

for a positive homogeneously presented cancellative tame monoid whose idealistic subsemigroup is finitely generated, the conjugacy problem in it is solvable.

2. POSITIVELY PRESENTED MONOID

In this section, we first recall from [B-S] some basic definitions and notations. Secondly, for a positive finitely presented group

$$G = \langle L \mid R \rangle,$$

we associate a monoid defined by it. We will extend a basic notion in [B-S], *fundamental element*, for a positive presented atomic monoid. For a positively presented atomic cancellative monoid, we will show some convenient propositions.⁴ Lastly, by using a fundamental element Δ in the associated monoid, we will discuss the conjugacy problem in the group $G = \langle L \mid R \rangle$.

Let L be a finite set. Let $F(L)$ be the free group generated by L , and let L^* be the free monoid generated by L inside $F(L)$. We call the elements of $F(L)$ *words* and the elements of L^* *positive words*. The empty word ε is the identity element of L^* . If two words A, B are identical letter by letter, we write $A \equiv B$. Let $G = \langle L \mid R \rangle$ be a positively presented group (i.e. the set R of relations consists of those of the form $R_i = S_i$ where R_i and S_i are positive words), where R is the set of relations. We often denote the images of the letters and words under the quotient homomorphism

$$F(L) \longrightarrow G$$

by the same symbols and the equivalence relation on elements A and B in G is denoted by $A = B$. Secondly, we recall some terminologies and concepts on a monoid M . An element $U \in M$ is said to *divide* $V \in M$ from the left (resp. right), and denoted by $U \mid_l V$ (resp. $U \mid_r V$), if there exists $W \in M$ such that $V = UW$ (resp. $V = WU$). We also say that V is *left-divisible* (resp. *right-divisible*) by U , or V is a *right-multiple* (resp. *left-multiple*) of U . We say that M satisfies *the left* (resp. *right*) *LCM condition*, if for any two elements U, V in M , there always exists their left (resp. right) least common multiple. We say that M satisfies *the left* (resp. *right*) *GCD condition*, if for any two elements U, V in M , there always exists their left (resp. right) greatest common divisor. Lastly, we consider four operations on the set of subsets of a monoid M . For a subset J of M , we put

$$\begin{aligned} \text{cm}_r(J) &:= \{u \in M \mid j \mid_l u, \forall j \in J\}, \\ \text{cd}_l(J) &:= \{u \in M \mid u \mid_l j, \forall j \in J\}, \\ \text{min}_r(J) &:= \{u \in J \mid \exists v \in J \text{ s.t. } v \mid_l u \Rightarrow v = u\}, \\ \text{max}_l(J) &:= \{u \in J \mid \exists v \in J \text{ s.t. } u \mid_l v \Rightarrow u = v\}. \end{aligned}$$

We note that $\text{cm}_r(J)$, $\text{min}_r(J)$, and $\text{max}_l(J)$ may be the empty set. We consider their compositions by

$$\begin{aligned} \text{mcm}_r(J) &:= \text{min}_r(\text{cm}_r(J)), \\ \text{mcd}_l(J) &:= \text{max}_l(\text{cd}_l(J)). \end{aligned}$$

In the same way, we define two operations $\text{mcm}_l(J)$ and $\text{mcd}_r(J)$. If M satisfies the right LCM condition, then we should write $\text{mcm}_r(J)$ by $\{\text{lcm}_r(J)\}$. If M satisfies the left GCD condition, then we should write $\text{mcd}_l(J)$ by $\{\text{gcd}_l(J)\}$. Next, we

⁴In this article, we only deal with the examples whose presentations are positive homogeneous. For some application, in this section we deal with a positively presented atomic monoid.

recall from [S-I], [I1], [I3] some terminologies and concepts on positively presented monoid. And we refer to some concepts from [D-P], [D1].

Definition 2.1. Let $G = \langle L \mid R \rangle$ be a positively finitely presented group, where L is the set of generators (called alphabet) and R is the set of relations. Then we associate a monoid $G^+ = \langle L \mid R \rangle_{mo}$ defined as the quotient of the free monoid L^* generated by L by the equivalence relation defined as follows:

i) two words U and V in L^* are called elementarily equivalent if either $U \equiv V$ or V is obtained from U by substituting a substring R_i of U by S_i where $R_i = S_i$ is a relation of R ($S_i = R_i$ is also a relation if $R_i = S_i$ is a relation),

ii) two words U and V in L^* are called equivalent, denoted by $U \doteq V$, if there exists a finite sequence $U \equiv W_0, W_1, \dots, W_n \equiv V$ of words in L^* for $n \in \mathbb{Z}_{\geq 0}$ such that W_i is elementarily equivalent to W_{i-1} for $i = 1, \dots, n$.

1. We say that G^+ is atomic, if there exists a map:

$$\nu : G^+ \longrightarrow \mathbb{Z}_{\geq 0}$$

such that i) $\nu(\alpha) = 0 \iff \alpha = 1$ and ii) an inequality:

$$\nu(\alpha\beta) \geq \nu(\alpha) + \nu(\beta)$$

is satisfied for any $\alpha, \beta \in G^+$. If $G^+ = \langle L \mid R \rangle_{mo}$ is a positive homogeneously presented monoid (i.e. the set R of relations consists of those of the form $R_i = S_i$ where R_i and S_i are positive words of the same length), it is clear that G^+ is an atomic monoid. An element $\alpha \neq 1$ in G^+ is called an atom if it is indecomposable, namely, $\alpha = \beta\gamma$ implies $\beta = 1$ or $\gamma = 1$.

2. We suppose that G^+ satisfies the condition of atomic monoid. Here, we write the set of generators L by $\{g_1, g_2, \dots, g_m\}$. If, for some positive word $w(g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_m)$ (i.e. a word that is written by the generators except g_i), $g_i = w(g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_m)$ is a relation of R , then we call the generator g_i a dummy generator. We note that, in the set R , a relation that has a form of $g_i = w(g_1, \dots, g_i, \dots, g_m)$ must be the form $g_i = w(g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_m)$ or a trivial form $g_i = g_i$, because we suppose here that G^+ is an atomic monoid. We denote by L' the set of all dummy generators of the monoid G^+ . We put $\tilde{L} := L \setminus L'$. We remark that the set \tilde{L} may be the empty set. We note that, if G^+ is an atomic monoid, the image of the set \tilde{L} in G^+ is equal to the set of all the atoms.

3. We say that G^+ is cancellative, if an equality $AXB \doteq AYB$ for $A, B, X, Y \in G^+$ implies $X \doteq Y$.

4. The natural homomorphism $\pi : G^+ \rightarrow G$ will be called the localization homomorphism.

5. An element $\Delta \in G^+$ is called quasi-central (also see [B-S] 7.1), if there exists a permutation σ of \tilde{L} such that

$$s \cdot \Delta \doteq \Delta \cdot \sigma(s)$$

holds for all generators $s \in \tilde{L}$. The set of all quasi-central elements is denoted by $\mathcal{QZ}(G^+)$. We note that, if the monoid G^+ is a cancellative monoid, there exists a unique permutation σ_Δ for a quasi-central element Δ . The order of an element σ_Δ in the permutation group $\mathfrak{S}(\tilde{L})$ is denoted by $\text{ord}(\sigma_\Delta)$. Note that $\Delta^{\text{ord}(\sigma_\Delta)}$ belongs to the center $\mathcal{Z}(G^+)$ of the monoid G^+ and $\varepsilon \in \mathcal{QZ}(G^+)$.

6. An element $\Delta \in G^+$ is called a Garside element if the sets of left- and right-divisors of Δ coincide, generate G^+ , and are finite in number.

7. An element Δ in G^+ is called a fundamental element if there exists a permutation σ of \tilde{L} such that, for any $s \in \tilde{L}$, there exists $\Delta_s \in G^+$ satisfying the following relation:

$$\Delta = s \cdot \Delta_s = \Delta_s \cdot \sigma(s).$$

We write the set of all fundamental elements of G^+ by $\mathcal{F}(G^+)$. Note that $\varepsilon \notin \mathcal{F}(G^+)$. If the monoid G^+ is a cancellative monoid, there exists a unique permutation σ_Δ for a fundamental element Δ . Note that $\Delta^{\text{ord}(\sigma_\Delta)}$ belongs to the center $\mathcal{Z}(G^+)$ of the monoid G^+ . It is easy to show that

$$\mathcal{F}(G^+) \mathcal{QZ}(G^+) = \mathcal{QZ}(G^+) \mathcal{F}(G^+) = \mathcal{F}(G^+).$$

8. A fundamental element Δ is called minimal, if any fundamental element dividing Δ from right or left coincides with Δ itself.

9. A quasi-central element Δ is called indecomposable, if it does not decompose into a product of two non-trivial quasi-central elements. We note that the identity element ε is not indecomposable.

Remark 1. Let G^+ be a positively presented atomic monoid $\langle L \mid R \rangle_{mo}$ that satisfies the cancellation condition. Let Δ be a non-trivial quasi-central element. We put

$$L(\Delta) := \{s \in \tilde{L} \mid s \mid_l \Delta\}.$$

Then, we have an equality

$$\forall s \in L(\Delta), \Delta = s \cdot \Delta_s = \Delta_s \cdot \sigma_\Delta(s).$$

For each s in $L(\Delta)$, the quotient can be uniquely determined in the monoid G^+ . Here, we write it by Δ_s . From this property, we easily show that the sets of left- and right-divisors of Δ coincide.

Remark 2. Let G^+ be a positively presented atomic monoid. The two-sided idealistic subsemigroup $\mathcal{F}(G^+)(\subseteq \mathcal{QZ}(G^+))$ is not finitely generated in general (See [II] Remark 5.9).

From the definitions, it follows that the notion of fundamental elements is equivalent to the notion of Garside elements.

Proposition 2.2. Let G^+ be a positively presented atomic monoid $\langle L \mid R \rangle_{mo}$ that satisfies the cancellation condition. Then:

(1) An element Δ in G^+ is a fundamental element if and only if Δ is a Garside element.

(2) Let Δ_1 and Δ_2 be quasi-central elements in G^+ that satisfies a relation: there exists an element d in G^+ such that $\Delta_1 d = \Delta_2$. Then, the element d is a quasi-central element in G^+

Proof. See [I3] §2. □

Proposition 2.3. Let G^+ be a positively presented atomic cancellative monoid $\langle L \mid R \rangle_{mo}$ that carries a fundamental element Δ and satisfies the condition that any two letters α, β in \tilde{L} admit the left (resp. right) least common multiple. Then, the monoid G^+ satisfies the left (resp. right) LCM condition.

Proof. Let $\Delta \in \mathcal{F}(G^+)$ be a fundamental element. Due to the property of fundamental element, for any $\gamma \in G^+$, there exists a sufficiently large integer l such that γ divides Δ^l from the left and the right. Therefore, we say that, for any two elements U, V in G^+ , $\text{mcm}_r(\{U, V\}) \neq \emptyset$. We consider the following equation

$$UX \doteq VY.$$

Since the monoid G^+ is cancellative and, for any two letters α, β in \tilde{L} , they admit the right least common multiple, we solve the equation uniquely, if the solution exists. The existence of the solution is guaranteed since, for any two elements U, V in G^+ , $\text{mcm}_r(\{U, V\}) \neq \emptyset$. We conclude that the monoid G^+ satisfies the right LCM condition. In the same way, we also conclude that the monoid G^+ satisfies the left LCM condition. \square

Proposition 2.4. Let G^+ be a positively presented atomic monoid $\langle L \mid R \rangle_{mo}$ that carries a fundamental element Δ .

Then, the monoid G^+ satisfies the left (resp. right) LCM condition if and only if the monoid G^+ satisfies the right (resp. left) GCD condition.

Proof. Assume that the monoid G^+ satisfies the left LCM condition. We easily show that, for any two elements α, β in G^+ , the element $\text{lcm}_l(\text{cd}_r(\{\alpha, \beta\}))$ is the right greatest common divisor for the set $\{\alpha, \beta\}$.

Next we assume the monoid G^+ satisfies the right GCD condition. We suppose that the monoid G^+ does not satisfy the left LCM condition. Then, there exist two elements α, β in G^+ such that the set $\text{mcm}_l(\{\alpha, \beta\})$ does not consist of a single element. Thanks to the existence of a fundamental element Δ , we say that $\text{mcm}_l(\{\alpha, \beta\}) \neq \emptyset$. Therefore, we can take two different elements C_1, C_2 from the set $\text{mcm}_l(\{\alpha, \beta\})$. We say that both α and β are left divisors of $\text{gcd}_r(\{C_1, C_2\})$. Hence, we say that

$$C_1 = \text{gcd}_r(\{C_1, C_2\}) = C_2.$$

We have a contradiction. \square

Proposition 2.5. Let G^+ be a positively presented atomic monoid $\langle L \mid R \rangle_{mo}$ that satisfies the cancellation condition. We suppose that the monoid G^+ satisfies the LCM condition and $\mathcal{F}(G^+) \neq \emptyset$ (i.e. the monoid G^+ is a Garside monoid).

Then, the idealistic subsemigroup $\mathcal{F}(G^+)(\subseteq \mathcal{QZ}(G^+))$ is singly generated.

Proof. We take arbitrary two elements Δ_1, Δ_2 in $\mathcal{F}(G^+)$. Due to the Proposition 2.4, we say that, for two elements Δ_1, Δ_2 , there exists the right (resp. left) greatest common divisor. Since both Δ_1 and Δ_2 are Garside elements, we easily show that

$$\gcd_r(\{\Delta_1, \Delta_2\}) = \gcd_l(\{\Delta_1, \Delta_2\}).$$

We put $\delta := \gcd_r(\{\Delta_1, \Delta_2\}) = \gcd_l(\{\Delta_1, \Delta_2\})$. We easily show that the element δ is a Garside element. We also say that the element δ is a fundamental element. Hence, there exists a unique element $\hat{\Delta}$ such that $\mathcal{F}(G^+)(\subseteq \mathcal{QZ}(G^+))$ is singly generated by $\hat{\Delta}$. \square

Thanks to the Proposition 2.5, in a Garside monoid we can choose the unique minimal fundamental element $\hat{\Delta}$.

Lemma 2.6. *Let G^+ be a positively presented atomic monoid $\langle L \mid R \rangle_{mo}$ that satisfies the cancellation condition and carries a fundamental element. Let δ be an element that satisfies the condition that there exists a mapping $\sigma : \tilde{L} \rightarrow G^+$*

$$\forall s \in \tilde{L}, s \cdot \delta = \delta \cdot \sigma(s).$$

Then, the element δ is a quasi-central element.

Proof. Due to the property of fundamental elements, there exists a fundamental element Δ such that

$$\Delta = \delta \cdot D.$$

In addition to this property, we can assume that, for any letter s in the set \tilde{L} , the letter s divides D from the left and the right. Since the monoid G^+ is a cancellative monoid, there exists a unique permutation σ_Δ for the fundamental element Δ . From the assumption, we have an equation

$$\forall s \in \tilde{L}, s \cdot \Delta = \delta \cdot \sigma(s) \cdot D = \delta \cdot D \cdot \sigma_\Delta(s).$$

Due to the cancellativity, we have an equation

$$(2.1) \quad \forall s \in \tilde{L}, \sigma(s) \cdot D = D \cdot \sigma_\Delta(s).$$

We put

$$L(\delta) := \{s \in \tilde{L} \mid s \mid_l \delta\}.$$

By using the equation (2.1), we also have an equation

$$(2.2) \quad \forall s \in L(\delta), s \cdot \delta_s = \delta_s \cdot \sigma(s).$$

For each s in $L(\delta)$, the quotient can be uniquely determined in the monoid G^+ . Here, we write it by δ_s . We put

$$L_{reg}(\delta) := \{s \in L(\delta) \mid \forall k \in \mathbb{Z}_{\geq 0}, \sigma^k(s) \in \tilde{L}\}, L_{irreg}(\delta) := L(\delta) \setminus L_{reg}(\delta).$$

Since the set \tilde{L} is a finite set, the restriction $\sigma|_{L_{reg}(\delta)}$ is a permutation.

Claim. $L_{reg}(\delta) \neq \emptyset$.

Proof. We assume that $L_{reg}(\delta) = \emptyset$. For any element s in $L(\delta)$, there exists a positive integer $k(s)$ such that

$$\sigma^{k(s)}(s) \in G^+ \setminus \tilde{L}.$$

We write the element δ by a product of atoms

$$\delta = \alpha_1 \cdot \alpha_2 \cdots \alpha_l.$$

From the equation (2.2), we have an equation

$$\begin{aligned}\alpha_1 \cdot \alpha_2 \cdots \alpha_l &= \sigma(\alpha_1) \cdot \sigma(\alpha_2) \cdots \sigma(\alpha_l) \\ \cdots &= \sigma^N(\alpha_1) \cdot \sigma^N(\alpha_2) \cdots \sigma^N(\alpha_l).\end{aligned}$$

Hence, expressions of the element δ by product of atoms do not have bounded lengths. Since the monoid G^+ is an atomic monoid, we have a contradiction. \square

We assume that $L_{irreg}(\delta) \neq \emptyset$. We put

$$L'_{irreg}(\delta) := \{s \in L_{irreg}(\delta) \mid \sigma(s) \in G^+ \setminus \tilde{L}\}.$$

We consider the following two cases.

Case 1: $\exists s_0 \in L'_{irreg}(\delta)$ s.t. the expressions of the element $\sigma(s_0)$ consist of atoms in $L_{reg}(\delta)$.

We write the element $\sigma(s_0)$ by a product of atoms in $L_{reg}(\delta)$

$$\sigma(s_0) = \beta_1 \cdot \beta_2 \cdots \beta_k.$$

Then, there exist atoms β'_i in $L_{reg}(\delta)$ ($i = 1, \dots, k$) such that $\sigma(\beta'_i) = \beta_i$ ($i = 1, \dots, k$). Therefore, we have an equation

$$(2.3) \quad \delta = \beta'_k \cdot \delta_{\beta'_k} = \delta_{\beta'_k} \cdot \beta_k = \delta_{s_0} \cdot \beta_1 \cdot \beta_2 \cdots \beta_k.$$

Due to the cancellativity, we have an equality

$$\delta_{\beta'_k} = \delta_{s_0} \cdot \beta_1 \cdot \beta_2 \cdots \beta_{k-1}.$$

By substituting $\delta_{\beta'_k}$ by $\delta_{s_0} \cdot \beta_1 \cdot \beta_2 \cdots \beta_{k-1}$ in the equation (2.3), we have

$$s_0 \cdot \delta_{s_0} = \delta = \beta'_k \cdot \delta_{s_0} \cdot \beta_1 \cdot \beta_2 \cdots \beta_{k-1} = \beta'_1 \cdot \beta'_2 \cdots \beta'_k \cdot \delta_{s_0}.$$

Then, we have an equality $s_0 = \beta'_1 \cdot \beta'_2 \cdots \beta'_k$. A contradiction.

Case 2: $\forall s \in L'_{irreg}(\delta)$, the expressions of the element $\sigma(s)$ by product of atoms contain an atom in $L_{irreg}(\delta)$.

We consider an equation again

$$\begin{aligned}\delta &= \alpha_1 \cdot \alpha_2 \cdots \alpha_l = \sigma(\alpha_1) \cdot \sigma(\alpha_2) \cdots \sigma(\alpha_l) \\ \cdots &= \sigma^N(\alpha_1) \cdot \sigma^N(\alpha_2) \cdots \sigma^N(\alpha_l).\end{aligned}$$

Since the monoid G^+ is an atomic monoid, we have a contradiction.

Therefore, we conclude that $L(\delta) = L_{reg}(\delta)$. Next, we put

$$L(D) := \{s \in \tilde{L} \mid s \mid_r D\}, L_{reg}(D) := \{s \in L(D) \mid \forall k \in \mathbb{Z}_{\geq 0}, \varphi^k(s) \in \tilde{L}\}$$

, where $\varphi(s) := \sigma(\sigma_{\Delta}^{-1}(s))$. In the same way, we conclude that $L(D) = L_{reg}(D)$. From the assumption, we say that $L(D) = \tilde{L}$. Hence, the element D is a fundamental element. We conclude that δ is a quasi-central element. \square

Lemma 2.7. We suppose that a positively presented monoid $G^+ = \langle L \mid R \rangle_{mo}$ is a Garside monoid. Let Δ_1 and Δ_2 be quasi-central elements in G^+ .

Then, the element $\gcd_l(\{\Delta_1, \Delta_2\}) = \gcd_r(\{\Delta_1, \Delta_2\})$ is a quasi-central element.

Proof. Since the monoid satisfies the LCM condition, the GCD condition is also satisfied. We put $\delta_l := \gcd_l(\{\Delta_1, \Delta_2\})$ and $\delta_r := \gcd_r(\{\Delta_1, \Delta_2\})$. Due to the Remark 1, we easily show that $\delta_l = \delta_r$. We write it by δ . If δ is equal to Δ_1 or Δ_2 , then δ is a quasi-central element. Therefore, we suppose that $\delta \neq \Delta_1, \Delta_2$. If the element δ satisfies the following condition

$$\forall s \in \tilde{L}, \text{lcm}_r(\{s \cdot \delta, \delta\}) = \{s \cdot \delta\},$$

due to the Lemma 2.6, we say that there exists a map $\sigma : \tilde{L} \rightarrow \tilde{L}$ such that

$$\forall s \in \tilde{L}, s \cdot \delta = \delta \cdot \sigma(s).$$

Since the monoid G^+ is a cancellative monoid, we say that the map σ is a permutation of \tilde{L} . Therefore, we say that δ is a quasi-central element. If the element δ does not satisfy the above condition, then we say that there exists a letter $s_0 \in \tilde{L}$ such that there exists a unique element $d \neq \varepsilon$ such that

$$\text{lcm}_r(\{s_0 \cdot \delta, \delta\}) = \{s_0 \cdot \delta d\}.$$

The uniqueness of d is guaranteed by the LCM condition. We define two elements X_1, X_2 by

$$\Delta_1 = \delta \cdot X_1, \Delta_2 = \delta \cdot X_2.$$

Since the monoid G^+ is a cancellative monoid, we say that two elements X_1 and X_2 are determined uniquely. We easily show that $d|_l X_1$ and $d|_l X_2$. Hence, we say that $\delta \cdot d \in \text{cd}_l(\{\Delta_1, \Delta_2\})$. We have a contradiction. Therefore, we conclude that the element δ is a quasi-central element. \square

As a consequence of Proposition 2.7, we prove the following proposition.

Proposition 2.8. We suppose that a positively presented monoid $G^+ = \langle L \mid R \rangle_{mo}$ is a Garside monoid. Let $\hat{\Delta}$ be the unique minimal fundamental element in G^+ , and let Δ be an indecomposable quasi-central element in G^+ . Then, the element Δ divides $\hat{\Delta}$ from the left and the right.

Definition 2.9. Let G^+ be a positively presented atomic cancellative monoid $\langle L \mid R \rangle_{mo}$ that carries a fundamental element.

The monoid G^+ is called tame if for any indecomposable quasi-central element Δ_0 in G^+ there exists a minimal fundamental element Δ such that the element Δ_0 divides Δ from the left and the right.

Remark 3. In [11], the author studied some positive homogeneously presented cancellative monoid. The author made a list of all the minimal fundamental elements and indecomposable quasi-central elements in the monoid. The monoid is not tame.

Lastly, we discuss the word and conjugacy problem in a positive homogeneously presented group.

Definition 2.10. Let $G^+ = \langle L \mid R \rangle_{mo}$ be a positive homogeneously presented monoid.

1) For arbitrary two words U, V in L^* , give an algorithm that decides whether $U = V$ in G^+ or not.

2) For arbitrary two words U, V in L^* , give an algorithm that decides whether there exists an element A in G^+ such that $AU = VA$ (then we write $U \sim_{mo} V$) or not.

The problems 1), 2) are called the word problem and the conjugacy problem in a monoid G^+ , respectively.

Remark 4. If $\mathcal{F}(G^+) \neq \emptyset$, then the relation \sim_{mo} is an equivalence relation.

Lemma 2.11. Let $G = \langle L \mid R \rangle$ be a positive homogeneously presented group, and let $G^+ = \langle L \mid R \rangle_{mo}$ be the associated monoid. Assume that the monoid G^+ is a cancellative monoid and $\mathcal{F}(G^+) \neq \emptyset$. Then:

- (1) The localization homomorphism $\pi : G^+ \rightarrow G$ is injective.
- (2) The word problem in G is solvable.
- (3) The conjugacy problem in G^+ is solvable if and only if the conjugacy problem in G is solvable.

Proof. (1) Let $\Delta \in \mathcal{F}(G^+)$ be a fundamental element. We can easily show that, for any $U \in G^+$, there exists a sufficiently large integer l such that U divides Δ^l from the left and the right. Hence, we show that the monoid G^+ satisfies Öre's condition (see [C-P]). Therefore, the localization homomorphism π is injective.

(2) We put $\Lambda := \Delta^{\text{ord}(\sigma_\Delta)}$, which belongs to the center $\mathcal{Z}(G^+)$ of the monoid G^+ . For any two elements U, V in G , there exists a non-negative integer k in $\mathbb{Z}_{\geq 0}$ such that both $(\pi(\Lambda))^k U$ and $(\pi(\Lambda))^k V$ are equivalent to positive words. Since the localization homomorphism π is injective, there exists a unique element $U' \in G^+$ (resp. $V' \in G^+$) such that

$$\pi(U') = (\pi(\Lambda))^k U \text{ (resp. } \pi(V') = (\pi(\Lambda))^k V \text{)}.$$

Therefore, we can show that $U = V$ can be shown in G algorithmically if and only if $U' = V'$ can be shown in G^+ algorithmically. Because the monoid G^+ is an atomic monoid, we can obtain algorithmically all the possible expressions of two words U' and V' in G^+ in a finite number of steps. Hence, by comparing two types of complete lists of all the possible expressions of words U' and V' , we decide in a finite number of steps whether $U' = V'$ or not. Consequently, the word problem in G can be solved.

(3) If two elements U and V in G are conjugate, then there exists a word B such that $BU = VB$. There exists a non-negative integer l in $\mathbb{Z}_{\geq 0}$ such that $(\pi(\Lambda))^l B$ is equivalent to a positive word. Since $\pi(\Lambda)$ belongs to the center of the group G , we say that two elements U and V in G are conjugate precisely when there is a positive word A such that AU is equivalent to VA . Therefore, due to the injectivity of the localization homomorphism π , we can show that the conjugacy problem in G^+ is solvable if and only if the conjugacy problem in G is solvable. \square

3. HOW TO SOLVE THE CONJUGACY PROBLEM

Let G^+ be a positive homogeneously presented cancellative monoid $\langle L \mid R \rangle_{mo}$ that carries a fundamental element Δ . In this section, we will discuss how to solve the conjugacy problem for G^+ by referring to the method given by E. Brieskorn and K. Saito in [B-S].

For an arbitrary element $w \in G^+$, we put

$$\text{Conj}^+(w) := \{V \in G^+ \mid AV = wA, A \in G^+\}.$$

If we give an algorithm to decide the set $\text{Conj}^+(w)$ for an arbitrary element $w \in G^+$, then we can say that the conjugacy problem for it is solvable. To decide the set

$\text{Conj}^+(w)$ for an arbitrary element $w \in G^+$, we introduce the following set

$$\text{O}^{(1)}(w; \Delta) := \{V \in G^+ \mid AV \doteq wA, A \mid_l \Delta\}.$$

Since the set of left divisors of Δ is a finite set, we say that one can decide the set $\text{O}^{(1)}(w; \Delta)$ algorithmically. Since the set $\text{O}^{(1)}(w; \Delta)$ is a finite set, one can iterate the construction and obtain the sets

$$\text{O}^{(k+1)}(w; \Delta) := \{V \in G^+ \mid AV \doteq UA, U \in \text{O}^{(k)}(w; \Delta), A \mid_l \Delta\}.$$

We easily show that the sets $\text{O}^{(k)}(w; \Delta)$ can be decided algorithmically. We have $\text{O}^{(k)}(w; \Delta) \subseteq \text{O}^{(k+1)}(w; \Delta)$. Due to the homogeneity, we say that all elements in $\text{O}^{(k)}(w; \Delta)$ have the same length. Hence, there exists a positive integer k_0 such that

$$\text{O}^{(k_0)}(w; \Delta) = \text{O}^{(k_0+1)}(w; \Delta) = \dots$$

We put

$$\text{O}(w; \Delta) := \text{O}^{(k_0)}(w; \Delta).$$

From the construction, we have $\text{O}(w; \Delta) \subseteq \text{Conj}^+(w)$.

Remark 5. If an equality $\text{O}(w; \Delta) = \text{Conj}^+(w)$ holds for an arbitrary element $w \in G^+$, then the conjugacy problem for it is solvable.

Definition 3.1. Let $G^+ = \langle L \mid R \rangle_{mo}$ be a positive homogeneously presented cancellative monoid that carries a fundamental element Δ , and let w be an element in G^+ .

1. An element A in G^+ is called a right transit element of w in G^+ , if
 - i) $A \neq \varepsilon$.
 - ii) There exists an element Q in G^+ such that an equation $AQ \doteq wA$ holds.
 The set of all right transit elements of w in G^+ shall be denoted by $\text{Trans}(w)$.
 2. A right transit element A of w is called minimal, if any right transit element of w dividing A from the left coincides with A itself.
- The set of all right minimal transit elements of w in G^+ shall be denoted by $\text{Trans}^{\min}(w)$.

Definition 3.2. Let $G^+ = \langle L \mid R \rangle_{mo}$ be a positive homogeneously presented cancellative monoid that carries a fundamental element Δ , and let w be an element in G^+ .

We introduce a property with respect to w :

$(P(w; \Delta))$: For any element A in $\text{Trans}(w)$, there exists an element $\delta(A)$ in $\text{Trans}(w)$ such that $\delta(A) \mid_l A$ and $\delta(A) \mid_l \Delta$ hold.

Lemma 3.3. Let $G^+ = \langle L \mid R \rangle_{mo}$ be a positive homogeneously presented cancellative monoid that carries a fundamental element Δ , and let w be an element in G^+ .

If the property $(P(w; \Delta))$ with respect to w is satisfied, then an equality

$$\text{O}(w; \Delta) = \text{Conj}^+(w)$$

holds.

Proof. We take an element V in $\text{Conj}^+(w)$. By definition, there exists an element A in G^+ such that an equality $AV \doteq wA$ holds. Since the property $(P(w; \Delta))$ is satisfied, there exists a sequence $(\delta_1, \delta_2, \dots, \delta_k)$ of elements in $\text{Trans}(w)$ such that each δ_i divides Δ ($i = 1, \dots, k$) from the left and we have a decomposition

$$A \doteq \delta_1 \cdot \delta_2 \cdots \delta_k.$$

Hence, we have $V \in \text{O}(w; \Delta)$. □

Theorem 3.4. Let $G^+ = \langle L \mid R \rangle_{mo}$ be a positive homogeneously presented cancellative monoid that carries a fundamental element Δ and satisfies the LCM condition. Then, the conjugacy problem for G^+ is solvable.

Proof. We take an arbitrary element w in G^+ . We verify the property $(P(w; \Delta))$ with respect to w . We take an element A in $\text{Trans}(w)$. By definition, there exists an element Q in G^+ such that an equation $AQ = wA$ holds. Since the element A is not ε , there exists a letter l_0 such that the letter l_0 divides A from the left. Then, there exists a unique element d_1 such that

$$\text{lcm}_r(\{l_0, w \cdot l_0\}) = \{w \cdot d_1\}.$$

Thanks to the property of fundamental elements, we say that the element d_1 divides Δ from the left. We rewrite $d_0 = l_0$. One can iterate the construction and obtain a sequence (d_0, d_1, d_2, \dots) of elements in G^+ . Namely, d_i and d_{i+1} satisfy the relation $\text{lcm}_r(\{d_i, w \cdot d_i\}) = \{w \cdot d_{i+1}\}$ for $i = 0, 1, \dots$. We easily show that the element d_i divides d_{i+1} from the left for $i = 0, 1, \dots$. Thanks to the property of fundamental elements, we also say that the element d_i divides Δ from the left for $i = 0, 1, \dots$. Due to the homogeneity of the defining relations in the monoid G^+ , there exists a positive integer k_0 such that

$$d_{k_0} = d_{k_0+1} = \dots$$

Then, the element d_{k_0} belongs to the set $\text{Trans}(w)$. Thanks to the Lemma 3.3, we say that the conjugacy problem for G^+ is solvable. \square

Remark 6. The element d_{k_0} is not a right minimal transit element of w in general.

Though there exists an element w_0 in G^+ such that the property $(P(w_0; \Delta))$ is not satisfied, there is a possibility for solving the conjugacy problem for G^+ . Due to the homogeneity, we show that for an arbitrary element w in G^+ the set $\text{Conj}^+(w)$ is a finite set. To solve the conjugacy problem for G^+ , it is sufficient to show that for an arbitrary element w in G^+ one can decide the set $\text{Trans}^{\min}(w)$ explicitly.

Example. We recall an example, the monoid G_{Bii}^+ , from [I1]. The monoid G_{Bii}^+ has the following presentation

$$G_{\text{Bii}}^+ := \left\langle a, b, c \mid \begin{array}{l} cbb = bba, \\ ab = bc, \\ ac = ca \end{array} \right\rangle_{mo}.$$

We recall a lemma from [S-I] §7.

Lemma 3.5. Let X and Y be two words in G_{Bii}^+ of length $r \in \mathbb{Z}_{\geq 0}$.

- (i) If $vX = vY$ for some $v \in \{a, b, c\}$, then $X = Y$.
- (ii) If $aX = bY$, then $X = bZ$, $Y = cZ$ for some positive word Z .
- (iii) If $aX = cY$, then $X = cZ$, $Y = aZ$ for some positive word Z .
- (iv) If $bX = cY$, then there exist an integer $k \in \mathbb{Z}_{\geq 0}$ and a positive word Z such that $X = c^k b a \cdot Z$, $Y = a^k b b \cdot Z$.

Thanks to the Lemma 3.5, we say that the monoid G_{Bii}^+ is a left cancellative monoid. In the monoid G_{Bii}^+ , we have an anti-homomorphism $\varphi : G_{\text{Bii}}^+ \rightarrow G_{\text{Bii}}^+$, $W \mapsto \varphi(W) := \sigma(\text{rev}(W))$, where σ is a permutation $\begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}$ and $\text{rev}(W)$ is the reverse of the word $W = x_1 x_2 \dots x_t$ (x_i is a letter or an inverse of a letter) given by

the word $x_t \cdots x_2 x_1$. If $\beta\alpha = \gamma\alpha$, then $\varphi(\beta\alpha) = \varphi(\gamma\alpha)$, i.e., $\varphi(\alpha)\varphi(\beta) = \varphi(\alpha)\varphi(\gamma)$. Using the left cancellation condition, we obtain $\varphi(\beta) = \varphi(\gamma)$ and, hence, $\beta = \gamma$.

From the Lemma 3.5 (iv), we say that in the monoid G_{Bii}^+ the LCM condition is not satisfied. Nevertheless, we can decide the set $\text{Trans}^{\min}(w)$ explicitly for an arbitrary element w in G_{Bii}^+ .

The monoid G_{Bii}^+ carries a unique indecomposable quasi-central element $\Delta_0 := bbb$ that is not a fundamental element and infinite minimal fundamental elements $\Delta_k := (bc^k)^3$. If Δ' is an indecomposable quasi-central element, then there exists a non-negative integer k in $\mathbb{Z}_{\geq 0}$ such that Δ' is equivalent to Δ_k . By using the element Δ_0 , we can introduce a special normal form in the monoid G_{Bii}^+ . Since both sides of the defining relations of the monoid G_{Bii}^+ contain the same number of the letter b , for an arbitrary element W in the monoid G_{Bii}^+ , the number of the letter b in W ought to be preserved in the process of rewriting W .

For each $j \in \mathbb{Z}_{\geq 0}$, let

$$W(j) := \{w \in G_{\text{Bii}}^+ \mid w \text{ contains the letter } b \text{ just } j\text{-times}\}.$$

We recall two facts from [I1] §5.

Proposition 3.6. *If $w \in W(j)$ ($j \geq 4$), then $\Delta_0 |_l w$ and $\Delta_0 |_r w$.*

Therefore, for an arbitrary element w in G_{Bii}^+ , we define a non-negative integer

$$k(w) := \max\{k \in \mathbb{Z}_{\geq 0} \mid \Delta_0^k |_l w\}.$$

Proposition 3.7. *Let w be an element in the monoid G_{Bii}^+ . We write*

$$w = \Delta_0^{k(w)} \cdot w_{\text{remain}}.$$

Let j be the number of the letter b in w_{remain} . Then, w_{remain} has the following normal form:

$$\begin{aligned} j = 0: & \quad w_{\text{remain}} = a^p c^q & (p, q \in \mathbb{Z}_{\geq 0}) \\ j = 1: & \quad w_{\text{remain}} = a^p c^q b a^r & (p, q, r \in \mathbb{Z}_{\geq 0}) \\ j = 2: & \quad w_{\text{remain}} = a^p c^q b b c^r & (p, q, r \in \mathbb{Z}_{\geq 0}) \\ j = 3: & \quad w_{\text{remain}} = a^p c^q b a^r b b & (p, q, r \in \mathbb{Z}_{\geq 0}) \end{aligned}$$

The element Δ_0 belongs to the center $\mathcal{Z}(G_{\text{Bii}}^+)$ of the monoid G_{Bii}^+ . To solve the conjugacy problem for G_{Bii}^+ , it is sufficient to show that, for an arbitrary element $w \in W(j)$ ($j \leq 3$), we decide the set $\text{Trans}^{\min}(w)$ explicitly. Here, we only show the results. The calculations by using the Lemma 3.5 are written in [I4]. We consider the following four cases.

Case 1. Let w be an element in $W(0)$.

$$p = 0, q \geq 1: \quad \text{Trans}^{\min}(w) = \{a, c, bb\}.$$

$$q = 0, p \geq 1: \quad \text{Trans}^{\min}(w) = \{a, b, c\}.$$

$$p, q \geq 1: \quad \text{Trans}^{\min}(w) = \{a, c\} \cup \{bbc^i b \mid i = 0, 1, \dots\}.$$

Since $\Delta_i = bc^i bc^i bc^i = bbc^i ba^i c^i$, we say that for any element u in $\text{Trans}^{\min}(w)$ there exists a non-negative integer $i \in \mathbb{Z}_{\geq 0}$ such that u divides Δ_i from the left.

Case 2. Let w be an element in $W(1)$.

$$p = 0, q \geq 1, r \geq 0: \quad \text{Trans}^{\min}(w) = \{b, c\}.$$

$$q = 0, p \geq 1, r \geq 0: \quad \text{Trans}^{\min}(w) = \{a, b, cba, \dots, c^r b a^r\}.$$

$$p, q \geq 1, r \geq 0: \quad \text{Trans}^{\min}(w) = \{a, b, c\}.$$

$$p = q = 0, r \geq 0: \quad \text{Trans}^{\min}(w) = \{b, cba, \dots, c^r b a^r\} \cup \{a^i c^i b a^i \mid i = r+1, \dots\}.$$

Since $\Delta_i = bc^i bc^i bc^i = c^i b a^i b b a^i = a^i c^i b a^i b b$, we say that for any element u in $\text{Trans}^{\min}(w)$ there exists a non-negative integer $i \in \mathbb{Z}_{\geq 0}$ such that u divides Δ_i from the left.

Case 3. Let w be an element in $W(2)$.

$p = 0, q \geq 1, r \geq 0$: $\text{Trans}^{\min}(w) = \{b, c\}$.

$q = 0, p \geq 1, r \geq 0$: $\text{Trans}^{\min}(w) = \{a, b\}$.

$p, q \geq 1, r \geq 0$: $\text{Trans}^{\min}(w) = \{a, b, c\}$.

$p = q = 0, r \geq 0$: $\text{Trans}^{\min}(w) = \{b, acba, a^2c^2ba^2, \dots\}$.

Since $\Delta_i = bc^i bc^i bc^i = a^i c^i b a^i b b$, we say that for any element u in $\text{Trans}^{\min}(w)$ there exists a non-negative integer $i \in \mathbb{Z}_{\geq 0}$ such that u divides Δ_i from the left.

Case 4. Let w be an element in $W(3)$.

We have $\text{Trans}^{\min}(w) = \{a, b, c\}$.

For an arbitrary element w in G_{Bii}^+ , we easily show that, for any element u in the set $\text{Trans}^{\min}(w)$, there exists a non-negative integer $i \in \mathbb{Z}_{\geq 0}$ such that u divides Δ_i from the left.

From the results on the monoid G_{Bii}^+ , we may say that there is a relation between the set $\text{Trans}^{\min}(w)$ and the set of all indecomposable quasi-central elements and all minimal fundamental elements.

Question 3.8. Let $G^+ = \langle L \mid R \rangle_{mo}$ be a positive homogeneously presented cancellative monoid that carries a fundamental element, and let w be an element in G^+ . Furthermore, we suppose that the monoid G^+ is tame and the idealistic subsemigroup $\mathcal{F}(G^+) (\subseteq \mathcal{QZ}(G^+))$ is finitely generated by $\Delta_1, \dots, \Delta_k$. We ask whether, for any element u in the set $\text{Trans}^{\min}(w)$, there exists a minimal fundamental element Δ_i such that u divides Δ_i from the left and the right. Then, from the set $\{\Delta_1, \dots, \Delta_k\}$, we construct a fundamental element $\hat{\Delta}$ algorithmically such that each $\Delta_j (j = 1, \dots, k)$ divides $\hat{\Delta}$ from the left and the right. It comes to the conclusion that an equality

$$O(w; \hat{\Delta}) = \text{Conj}^+(w)$$

holds.

Example. In [D4], the author investigated the monoids M_1, M_2 and M_3 . All of them are positive homogeneously presented cancellative monoids that carry fundamental elements. We can easily show that they are tame and their idealistic subsemigroups are finitely generated. In each example, we can check the property $(P(w; \hat{\Delta}))$ for an arbitrary element w in the monoid. It comes to the conclusion that the equality $O(w; \hat{\Delta}) = \text{Conj}^+(w)$ holds.

4. THE CONJUGACY PROBLEM FOR THE MONOID $G_{m,n}^+$

In this section, we will deal with the examples $G_{m,n}^+$ ([I3]). For the monoids $G_{m,n}^+$, we will show that the idealistic subsemigroup is singly generated and they are tame monoids. However, they do not satisfy the LCM condition. Nevertheless, in the monoids $G_{m,n}^+$, we will show that the conjugacy problem can be solved by verifying the property $(P(w; \Delta))$ for an arbitrary element w in them.

First, we recall an example, the monoid $G_{m,n}^+$ ($m, n \in \{2, 3, \dots\}$), from [I3]. The author studied the presentation of the fundamental group of the complement of certain complexified real affine line arrangement and associated a monoid defined by it. The monoid has the following presentation:

$$\left\langle s, t_1, \dots, t_m, u_1, \dots, u_n \left| \begin{array}{l} [s, t_1, \dots, t_m], [s, u_1, \dots, u_n], \\ [t_i, u_j] \ (i = 1, \dots, m, j = 1, \dots, n) \end{array} \right. \right\rangle_{mo},$$

where a symbol $[x_{i_1}, x_{i_2}, \dots, x_{i_k}]$ denotes the cyclic relations:

$$x_{i_1} x_{i_2} \cdots x_{i_k} = x_{i_2} \cdots x_{i_k} x_{i_1} = x_{i_k} x_{i_1} \cdots x_{i_{k-1}}.$$

The monoid $G_{m,n}^+$ denotes the associated monoid. Before continuing further, we recall some notation from [I3] §4. We put

$$\begin{aligned} \Delta &:= s \cdot t_1 \cdots t_m \cdot u_1 \cdots u_n, \quad \Delta_1 := s \cdot t_1 \cdots t_m, \quad \Delta_2 := s \cdot u_1 \cdots u_n, \\ I_1 &:= \{1, \dots, m\}, \quad I_2 := \{1, \dots, n\}, \\ L_0 &:= \{s, t_1, \dots, t_m, u_1, \dots, u_n\}, \quad L_1 := \{t_1, \dots, t_m\}, \quad L_2 := \{u_1, \dots, u_n\}, \\ F_1^+ &:= F^+(\underline{t}), \quad F_2^+ := F^+(\underline{u}), \\ F_{1,\text{rm}}^+ &:= \{w(\underline{t}) \in F_1^+ \mid (t_1 \cdots t_m) \not\ll_r w(\underline{t})\}, \\ F_{2,\text{rm}}^+ &:= \{w(\underline{u}) \in F_2^+ \mid (u_1 \cdots u_n) \not\ll_r w(\underline{u})\}, \\ F_{1,\text{cons}}^+ &:= \{w \in F_1^+ \mid \exists i_0, j_0 \in I_1 (i_0 \leq j_0) \text{ s.t. } w = t_{i_0} t_{i_0+1} \cdots t_{j_0}\}, \\ F_{2,\text{cons}}^+ &:= \{w \in F_2^+ \mid \exists i_0, j_0 \in I_2 (i_0 \leq j_0) \text{ s.t. } w = u_{i_0} u_{i_0+1} \cdots u_{j_0}\}. \end{aligned}$$

For arbitrary element $w(\underline{t})$ in F_1^+ and $w(\underline{u})$ in F_2^+ , we put

$$\text{Div}_1(w(\underline{t})) := \{w \in F_{1,\text{cons}}^+ \mid w \mid_r w(\underline{t})\},$$

$$\text{Div}_2(w(\underline{u})) := \{w \in F_{2,\text{cons}}^+ \mid w \mid_r w(\underline{u})\}.$$

We remark that there exists a unique element $w_{0,1}$ in $\text{Div}_1(w(\underline{t}))$ (resp. $w_{0,2}$ in $\text{Div}_2(w(\underline{u}))$) such that $w_1 \mid_r w_{0,1}$ for any element w_1 in $\text{Div}_1(w(\underline{t}))$ (resp. $w_2 \mid_r w_{0,2}$ for any element w_2 in $\text{Div}_2(w(\underline{u}))$). We put

$$C_1(w(\underline{t})) := w_{0,1}, \quad C_2(w(\underline{u})) := w_{0,2}.$$

In view of the defining relations of $G_{m,n}^+$, there exists an element $w'(\underline{t})$ in F_1^+ (resp. $w'(\underline{u})$ in F_2^+) such that we have a decomposition $w(\underline{t}) \equiv w'(\underline{t})C_1(w(\underline{t}))$ (resp. $w(\underline{u}) \equiv w'(\underline{u})C_2(w(\underline{u}))$) in $G_{m,n}^+$. We put

$$R_1(w(\underline{t})) := w'(\underline{t}), \quad R_2(w(\underline{u})) := w'(\underline{u}).$$

For arbitrary left divisor v_1 of Δ_1 (resp. v_2 of Δ_2), the quotient can be uniquely determined in the monoid $G_{m,n}^+$ respectively. We denote it by Δ_{1,v_1} (resp. Δ_{2,v_2}). We recall a lemma from [I3] §4.

Lemma 4.1. Let X and Y be positive words in $G_{m,n}^+$ of length $r \in \mathbb{Z}_{\geq 0}$ and let $Y^{(h)}$ be a positive word in $G_{m,n}^+$ of length $h \in \{0, \dots, r\}$.

- (i) If $vX \equiv vY$ for some $v \in L_0$, then $X \equiv Y$.
- (ii) If $t_i X \equiv u_j Y$ ($t_i \in L_1, u_j \in L_2$), then $X \equiv u_j Z$, $Y \equiv t_i Z$ for some positive word Z .
- (iii) If $sX \equiv w(\underline{t})Y^{(h)}$ for some positive word $w(\underline{t})$ of length $r - h + 1$ in F_1^+ , then $X \equiv \Delta_{1,s} \cdot R_1(w(\underline{t})) \cdot Z$, $Y^{(h)} \equiv \Delta_{1,C_1(w(\underline{t}))} \cdot Z$ for some positive word Z .
- (iv) If $sX \equiv w(\underline{u})Y^{(h)}$ for some positive word $w(\underline{u})$ of length $r - h + 1$ in F_2^+ , then $X \equiv \Delta_{2,s} \cdot R_2(w(\underline{u})) \cdot Z$, $Y^{(h)} \equiv \Delta_{2,C_2(w(\underline{u}))} \cdot Z$ for some positive word Z .
- (v) If $t_i X \equiv w(\underline{t})Y^{(h)}$ for some t_i in L_1 and some positive word $w(\underline{t})$ of length $r - h + 1$ in F_1^+ that satisfies $t_i \not\ll_l w(\underline{t})$, then there exists a word $w(\underline{u})$ in $F_{2,\text{rm}}^+$ such

that $X = w(\underline{u}) \cdot \Delta_{1,t_i} \cdot R_1(w(\underline{t})) \cdot Z$, $Y^{(h)} = w(\underline{u}) \cdot \Delta_{1,C_1(w(\underline{t}))} \cdot Z$ for some positive word Z .

(vi) If $u_i X = w(\underline{u}) Y^{(h)}$ for some u_i in L_2 and some positive word $w(\underline{u})$ of length $r - h + 1$ in F_2^+ that satisfies $u_i \not\vdash_l w(\underline{u})$, then there exists a word $w(\underline{t})$ in $F_{1,rm}^+$ such that $X = w(\underline{t}) \cdot \Delta_{2,u_i} \cdot R_2(w(\underline{u})) \cdot Z$, $Y^{(h)} = w(\underline{t}) \cdot \Delta_{2,C_2(w(\underline{u}))} \cdot Z$ for some positive word Z .

Thanks to the Lemma 4.1, we say that the monoid $G_{m,n}^+$ is a left cancellative monoid. In the monoid $G_{m,n}^+$, we have an anti-homomorphism $\varphi : G_{m,n}^+ \rightarrow G_{m,n}^+$, $W \mapsto \varphi(W) := \sigma(\text{rev}(W))$, where σ is a permutation $\begin{pmatrix} s & t_1 & \cdots & t_m & u_1 & \cdots & u_n \\ s & t_m & \cdots & t_1 & u_n & \cdots & u_1 \end{pmatrix}$ and $\text{rev}(W)$ is the reverse of the word $W = x_1 x_2 \cdots x_k$ (x_i is a letter) given by the word $x_k \cdots x_2 x_1$. By a similar argument in Example §3, we show that the monoid $G_{m,n}^+$ is a right cancellative monoid.

We easily show that Δ is a fundamental element. We recall two facts from [I3] §5.

Proposition 4.2. *The element Δ is a unique minimal fundamental element and $\mathcal{F}(G_{m,n}^+) = \mathcal{QZ}(G_{m,n}^+) \setminus \{\varepsilon\}$.*

Proposition 4.3. $\text{mcm}_r(L_0) = \text{mcm}_l(L_0) = \{\Delta\}$.

Thanks to the Proposition 4.2, we say that the idealistic subsemigroup $\mathcal{F}(G_{m,n}^+) (\subseteq \mathcal{QZ}(G_{m,n}^+))$ is singly generated by Δ and the monoid $G_{m,n}^+$ is tame.

From the Lemma 4.1 (v) and (vi), we say that in the monoid $G_{m,n}^+$ the LCM condition is not satisfied. Nevertheless, we can verify the property $(P(w; \Delta))$ for an arbitrary element w in the monoid $G_{m,n}^+$.

We have an important remark on the monoid $G_{m,n}^+$.

Remark 7. *For each letter v in L_0 , both sides of the defining relations of $G_{m,n}^+$ contain the same number of the letter v . For arbitrary word W in $G_{m,n}^+$, the number of the letter v in W ought to be preserved in the process of rewriting W .*

We consider the following set

$$\mathcal{W}_{m,n} := \{w \in G_{m,n}^+ \mid w \text{ does not contain } \Delta_1 \text{ (resp. } \Delta_2) \text{ as substring}\}.$$

Proposition 4.4. *Let w be an element in the set $G_{m,n}^+$. If neither $\Delta_{1,s}$ nor $\Delta_{2,s}$ divides w from the left, then the element w belongs to the set $\mathcal{W}_{m,n}$.*

By definition, in the process of rewriting w in $\mathcal{W}_{m,n}$, we only use the defining relations $[t_i, u_j]$ ($i = 1, \dots, m, j = 1, \dots, n$) in $G_{m,n}^+$.

Proposition 4.5. *Let w be an element in $\mathcal{W}_{m,n}$. Then, the element w has the following normal form:*

the element w can be uniquely written in the form

$$w = w_0(\underline{t}) \cdot w_0(\underline{u}) \cdot s \cdot w_1(\underline{t}) \cdot w_1(\underline{u}) \cdot s \cdots s \cdot w_N(\underline{t}) \cdot w_N(\underline{u}),$$

where $w_0(\underline{t}), w_1(\underline{t}), \dots$, and $w_N(\underline{t})$ are some words in F_1^+ and $w_0(\underline{u}), w_1(\underline{u}), \dots$ and $w_N(\underline{u})$ are some words in F_2^+ .

Before continuing further, we prepare some notation. For an element $w(\underline{t}) \in F_1^+$ (resp. $w(\underline{u}) \in F_2^+$), we define respectively

$$\tilde{C}_1(w(\underline{t}) \cdot s) := \begin{cases} C_1(w(\underline{t})) \cdot s & \text{if } t_m \mid_r C_1(w(\underline{t})) \\ s & \text{if } t_m \not\mid_r C_1(w(\underline{t})) \end{cases}$$

$$\tilde{C}_2(w(\underline{u}) \cdot s) := \begin{cases} C_2(w(\underline{u})) \cdot s & \text{if } u_n \mid_r C_2(w(\underline{u})) \\ s & \text{if } u_n \not\mid_r C_2(w(\underline{u})) \end{cases}$$

For the element $\tilde{C}_1(w(\underline{t}) \cdot s)$ (resp. $\tilde{C}_2(w(\underline{u}) \cdot s)$), the quotient can be uniquely determined respectively. We denote it by $\tilde{R}_1(w(\underline{t}) \cdot s)$ (resp. $\tilde{R}_2(w(\underline{u}) \cdot s)$). For an arbitrary element w in the monoid $G_{m,n}^+$, we define the set

$$L(w) := \{l \in L_0 \mid l \mid_l w\}.$$

Lemma 4.6. *Let t_i be a letter in L_1 . Let $w_0(\underline{t})$ be an element in $F_{1,\text{rm}}^+$ that satisfies $t_i \not\mid_l w_0(\underline{t})$, and let $w_0(\underline{u})$ be an element in $F_{2,\text{rm}}^+$.*

Then, for an equation $t_i \cdot X = w_0(\underline{t}) \cdot w_0(\underline{u}) \cdot s \cdot Y$, there exists a positive word Z such that

$$X = w_0(\underline{u}) \cdot \Delta_{1,t_i} \cdot \tilde{R}_1(w_0(\underline{t}) \cdot s) \cdot Z \text{ and } Y = \Delta_{1,\tilde{C}_1(w_0(\underline{t}) \cdot s)} \cdot Z.$$

Proof. Due to the Lemma 4.1, there exist $w_1(\underline{u})$ in $F_{2,\text{rm}}^+$ and a word Z_1 such that

$$X = w_0(\underline{u}) \cdot w_1(\underline{u}) \cdot \Delta_{1,t_i} \cdot R_1(w_0(\underline{t})) \cdot Z_1 \text{ and } s \cdot Y = w_1(\underline{u}) \cdot \Delta_{1,C_1(w_0(\underline{t}))} \cdot Z_1.$$

We consider the case $w_1(\underline{u}) \neq \varepsilon$. Then, there exists a word Z_2 such that $Y = \Delta_{2,s} \cdot R_2(w_1(\underline{u})) \cdot Z_2$. Thus, we have an equation

$$\Delta_2 \cdot R_2(w_1(\underline{u})) \cdot Z_2 = w_1(\underline{u}) \cdot \Delta_{1,C_1(w_0(\underline{t}))} \cdot Z_1.$$

By deviding the common factor $w_1(\underline{u})$ from the left, we have an equation

$$(4.1) \quad \Delta_{2,C_2(w_1(\underline{u}))} \cdot Z_2 = \Delta_{1,C_1(w_0(\underline{t}))} \cdot Z_1.$$

We can find a general solution of the equation (4.1)

$$Z_1 = \Delta_{2,s} \cdot C_1(w_0(\underline{t})) \cdot Z_3, \quad Z_2 = \Delta_{1,s} \cdot C_2(w_1(\underline{u})) \cdot Z_4.$$

Thus, we have

$$\begin{aligned} X &= w_0(\underline{u}) \cdot w_1(\underline{u}) \cdot \Delta_{1,t_i} \cdot R_1(w_0(\underline{t})) \cdot \Delta_{2,s} \cdot C_1(w_0(\underline{t})) \cdot Z_3 \\ &= w_0(\underline{u}) \cdot \Delta_{1,t_i} \cdot w_0(\underline{t}) \cdot \Delta_{2,s} \cdot w_1(\underline{u}), \\ Y &= \Delta_{2,s} \cdot R_2(w_1(\underline{u})) \cdot \Delta_{1,s} \cdot C_2(w_1(\underline{u})) \cdot Z_4 = \Delta_{1,s} \cdot \Delta_{2,s} \cdot w_1(\underline{u}). \end{aligned}$$

This completes the proof. \square

Lemma 4.7. *Let t_i be a letter in L_1 , and let w be an element in $\mathcal{W}_{m,n}$ that contains at least one letter s and satisfies $t_i \not\mid_l w$. We suppose that the element w has the normal form: $w_0(\underline{t}) \cdot w_0(\underline{u}) \cdot s \cdot w_1(\underline{t}) \cdot w_1(\underline{u}) \cdot s \cdots s \cdot w_N(\underline{t}) \cdot w_N(\underline{u}) \cdot s$.*

Then, for an equation $t_i \cdot X = w \cdot Y$, there exists a positive word Z such that

$$\begin{aligned} X &= w_0(\underline{u}) \cdot \Delta_{1,t_i} \cdot w_0(\underline{t}) \cdot w_1(\underline{u}) \cdot s \cdots s \cdot w_{N-1}(\underline{t}) \cdot w_N(\underline{u}) \cdot s \cdot \tilde{R}_1(w_N(\underline{t})s) \cdot Z, \\ Y &= \Delta_{1,\tilde{C}_1(w_N(\underline{t}) \cdot s)} \cdot Z. \end{aligned}$$

Proof. By applying the Lemma 4.6 to the equation $t_i \cdot X = w \cdot Y$ repeatedly, we say that the element $\tilde{C}_1(w_N(\underline{t}) \cdot s)$ divides Y from the left. Then, we write $Y = \Delta_{1,\tilde{C}_1(w_N(\underline{t}) \cdot s)} \cdot Z$ by some Z . By substituting this for the equation $t_i \cdot X = w \cdot Y$, we have the result. This completes the proof. \square

Lemma 4.8. Let t_i be a letter in L_1 , and let w be an element in $\mathcal{W}_{m,n}$ that contains at least one letter s and satisfies $t_i \nmid_l w$. We suppose that the element w has the normal form: $w_0(\underline{t}) \cdot w_0(\underline{u}) \cdot s \cdot w_1(\underline{t}) \cdot w_1(\underline{u}) \cdot s \cdots s \cdot w_N(\underline{t}) \cdot w_N(\underline{u})$.

Then, $\text{mcm}_r(\{t_i, w\})$

$$= \begin{cases} \{w \cdot \Delta_{1, \tilde{C}_1(w_{N-1}(\underline{t}) \cdot s) \cdot w_N(\underline{t})}\} & \text{if } w_N(\underline{t}) \mid_l \Delta_{1, \tilde{C}_1(w_{N-1}(\underline{t}) \cdot s)} \\ \{w \cdot w'(\underline{u}) \cdot \Delta_{1, C_1(w_N(\underline{t}))} \mid w'(\underline{u}) \in F_{2, \text{rm}}^+\} & \text{if } w_N(\underline{t}) \nmid_l \Delta_{1, \tilde{C}_1(w_{N-1}(\underline{t}) \cdot s)} \end{cases}$$

Proof. We put $w' = w_0(\underline{t}) \cdot w_0(\underline{u}) \cdot s \cdots s \cdot w_{N-1}(\underline{t}) \cdot w_{N-1}(\underline{u}) \cdot s$. We consider an equation $t_i \cdot X = w' \cdot Y$. Due to the Lemma 4.7, we say there exists a word Z_1 such that

$$Y = \Delta_{1, \tilde{C}_1(w_{N-1}(\underline{t}) \cdot s)} \cdot Z_1.$$

We consider an equation

$$w_N(\underline{t}) \cdot w_N(\underline{u}) \cdot Y' = \Delta_{1, \tilde{C}_1(w_{N-1}(\underline{t}) \cdot s)} \cdot Z_1.$$

If $w_N(\underline{t})$ divides the element $\Delta_{1, \tilde{C}_1(w_{N-1}(\underline{t}) \cdot s)}$ from the left, then due to the Lemma 4.1 we say that there exists a word Z_2 such that

$$Y' = \Delta_{1, \tilde{C}_1(w_{N-1}(\underline{t}) \cdot s) \cdot w_N(\underline{t})} \cdot Z_2.$$

If $w_N(\underline{t})$ does not divide the element $\Delta_{1, \tilde{C}_1(w_{N-1}(\underline{t}) \cdot s)}$, then due to the Lemma 4.1 we say that there exist a word $w'_0(\underline{u})$ in $F_{2, \text{rm}}^+$ and a word Z_3 such that

$$Y' = w'(\underline{u}) \cdot \Delta_{1, C_1(w_N(\underline{t}))}.$$

This completes the proof. \square

Lemma 4.9. Let u_i be a letter in L_2 , and let w be an element in $\mathcal{W}_{m,n}$ that contains at least one letter s and satisfies $u_i \nmid_l w$. We suppose that the element w has the normal form: $w_0(\underline{t}) \cdot w_0(\underline{u}) \cdot s \cdot w_1(\underline{t}) \cdot w_1(\underline{u}) \cdot s \cdots s \cdot w_N(\underline{t}) \cdot w_N(\underline{u})$.

Then, $\text{mcm}_r(\{u_i, w\})$

$$= \begin{cases} \{w \cdot \Delta_{2, \tilde{C}_2(w_{N-1}(\underline{u}) \cdot s) \cdot w_N(\underline{u})}\} & \text{if } w_N(\underline{u}) \mid_l \Delta_{2, \tilde{C}_2(w_{N-1}(\underline{u}) \cdot s)} \\ \{w \cdot w'(\underline{t}) \cdot \Delta_{2, C_2(w_N(\underline{u}))} \mid w'(\underline{t}) \in F_{1, \text{rm}}^+\} & \text{if } w_N(\underline{u}) \nmid_l \Delta_{2, \tilde{C}_2(w_{N-1}(\underline{u}) \cdot s)} \end{cases}$$

Lemma 4.10. Let w be an element in $\mathcal{W}_{m,n}$ that satisfies $s \nmid_l w$. We suppose that the element w has the normal form: $w_0(\underline{t}) \cdot w_0(\underline{u}) \cdot s \cdot w_1(\underline{t}) \cdot w_1(\underline{u}) \cdot s \cdots s \cdot w_N(\underline{t}) \cdot w_N(\underline{u})$.

Then, $\text{mcm}_r(\{s, w \cdot s\})$

$$= \begin{cases} \{w \cdot s \cdot \Delta_{1, \tilde{C}_1(w_N(\underline{t}) \cdot s)}\} & \text{if } w_0(\underline{t}) \neq \varepsilon, w_0(\underline{u}) = \varepsilon \\ \{w \cdot s \cdot \Delta_{2, \tilde{C}_2(w_N(\underline{u}) \cdot s)}\} & \text{if } w_0(\underline{t}) = \varepsilon, w_0(\underline{u}) \neq \varepsilon \\ \{w \cdot s \cdot \Delta_{1, \tilde{C}_1(w_N(\underline{t}) \cdot s)} \cdot \Delta_{2, \tilde{C}_2(w_N(\underline{u}) \cdot s)}\} & \text{if } w_0(\underline{t}) \neq \varepsilon, w_0(\underline{u}) \neq \varepsilon \end{cases}$$

For an arbitrary element w in $G_{m,n}^+$, we define a non-negative integer

$$k(w) := \max\{k \in \mathbb{Z}_{\geq 0} \mid \Delta^k \mid_l w\}.$$

We remark that one can decide the integer $k(w)$ algorithmically. We write

$$w = \Delta^{k(w)} \cdot w_{\text{remain}}.$$

If $k(w) > 0$, we have $L(w) = L_0$. Then, we easily verify the property $(P(w; \Delta))$ with respect to w . From here, we consider the case $k(w) = 0$. For the element w_{remain} and an index $i \in \{1, 2\}$, we define a non-negative integer

$$\lambda_i(w_{\text{remain}}) := \max\{k \in \mathbb{Z}_{\geq 0} \mid \Delta_i^k \mid_l w_{\text{remain}}\}.$$

By definition, we note that $\lambda_1(w_{\text{remain}}) \cdot \lambda_2(w_{\text{remain}}) = 0$. Hence, we consider the following three cases:

I: $\lambda_1(w_{\text{remain}}) > 0$, $\lambda_2(w_{\text{remain}}) = 0$.

For the element w_{remain} , we define a non-negative integer

$$\mu_1(w_{\text{remain}}) := \max\{k \in \mathbb{Z}_{\geq 0} \mid \Delta_1^{\lambda_1(w_{\text{remain}})} \cdot \Delta_{1,s}^k \mid_l w_{\text{remain}}\}.$$

We write

$$w \doteq w_{\text{remain}} \doteq \Delta_1^{\lambda_1(w_{\text{remain}})} \cdot \Delta_{1,s}^{\mu_1(w_{\text{remain}})} \cdot w'_{\text{remain}}.$$

From the Proposition 4.4, we say that the element w'_{remain} belongs to the set $\mathcal{W}_{m,n}$. We take an element A in $\text{Trans}(w)$. By definition, there exists an element Q in $G_{m,n}^+$ such that an equation $AQ \doteq wA$ holds. Since the element A is not ε , there exists a letter l_0 in L_0 such that the letter l_0 divides A from the left. We write $A \doteq l_0 \cdot A'$. Since $L(w) = \{s, t_1, \dots, t_m\}$, we have $\{s, t_1, \dots, t_m\} \subseteq \text{Trans}^{\min}(w)$. We should consider the case $l_0 = u_\tau$ ($u_\tau \in L_2$). We consider the following two cases.

Case 1. The element w'_{remain} does not contain the letter s .

We can write w'_{remain} by $w_0(\underline{t}) \cdot w_0(\underline{u})$. We consider an equation

$$u_\tau \cdot A' \cdot Q \doteq \Delta_1^{\lambda_1(w_{\text{remain}})} \cdot \Delta_{1,s}^{\mu_1(w_{\text{remain}})} \cdot w_0(\underline{t}) \cdot w_0(\underline{u}) \cdot A.$$

From the Lemma 4.1, there exists a word Z_1 such that

$$w_0(\underline{u}) \cdot A \doteq \Delta_{2,s} \cdot Z_1.$$

Due to the Lemma 4.1, there exist $w'_0(\underline{t})$ in $F_{1,\text{rm}}^+$ and a word Z_2 such that

$$A \doteq w'_0(\underline{t}) \cdot \Delta_{2,C_2(w_0(\underline{u}))} \cdot Z_2.$$

Since $\{s, t_1, \dots, t_m\} \subseteq \text{Trans}^{\min}(w)$, we should consider $w'_0(\underline{t}) = \varepsilon$. We easily show that $\{s, t_1, \dots, t_m, \Delta_{2,C_2(w_0(\underline{u}))}\} = \text{Trans}^{\min}(w)$. We have verified the property $(P(w; \Delta))$.

Case 2. The element w'_{remain} contains the letter s .

We write $w'_{\text{remain}} \doteq w_0(\underline{t}) \cdot w_0(\underline{u}) \cdot s \cdot w_1(\underline{t}) \cdot w_1(\underline{u}) \cdot s \cdots s \cdot w_N(\underline{t}) \cdot w_N(\underline{u})$. We consider an equation

$$u_\tau \cdot A' \cdot Q \doteq \Delta_1^{\lambda_1(w_{\text{remain}})} \cdot \Delta_{1,s}^{\mu_1(w_{\text{remain}})} \cdot w'_{\text{remain}} \cdot A.$$

From the Lemma 4.1, there exists a word Z_1 such that

$$w'_{\text{remain}} \cdot A \doteq \Delta_{2,s} \cdot Z_1.$$

Due to the Lemma 4.9, we consider the following two cases.

Case 2 – 1. $w_N(\underline{u}) \mid_l \Delta_{2,\tilde{C}_2(w_{N-1}(\underline{u}) \cdot s)}$

We say that there exists a word Z_1 such that

$$A \doteq \Delta_{2,\tilde{C}_2(w_{N-1}(\underline{u}) \cdot s) \cdot w_N(\underline{u})} \cdot Z_1.$$

We say that the element $\Delta_{2,\tilde{C}_2(w_{N-1}(\underline{u}) \cdot s) \cdot w_N(\underline{u})}$ belongs to the set $\text{Trans}(w)$. Since the element $\Delta_{2,\tilde{C}_2(w_{N-1}(\underline{u}) \cdot s) \cdot w_N(\underline{u})}$ divides Δ from the left, we have verified the property $(P(w; \Delta))$.

Case 2 – 2. $w_N(\underline{u}) \not\mid_l \Delta_{2,\tilde{C}_2(w_{N-1}(\underline{u}) \cdot s)}$

We say that there exist $w'_0(\underline{t})$ in $F_{1,\text{rm}}^+$ and a word Z_1 such that

$$A \doteq w'_0(\underline{t}) \cdot \Delta_{2,C_2(w_N(\underline{u}))} \cdot Z_1.$$

Since $\{s, t_1, \dots, t_m\} \subseteq \text{Trans}^{\min}(w)$, we should consider $w'_0(\underline{t}) = \varepsilon$. We say that the element $\Delta_{2, C_2(w_N(\underline{u}))}$ belongs to the set $\text{Trans}(w)$. Since the element $\Delta_{2, C_2(w_N(\underline{u}))}$ divides Δ from the left, we have verified the property $(P(w; \Delta))$.

II: $\lambda_1(w_{\text{remain}}) = 0, \lambda_2(w_{\text{remain}}) > 0$.

We can verify the property $(P(w; \Delta))$ in the same way as the case I.

III: $\lambda_1(w_{\text{remain}}) = \lambda_2(w_{\text{remain}}) = 0$.

For the element w_{remain} , we define a non-negative integer

$$\mu_1(w_{\text{remain}}) := \max\{k \in \mathbb{Z}_{\geq 0} \mid \Delta_{1,s}^k \mid w_{\text{remain}}\}.$$

Next, for the element w_{remain} , we define a non-negative integer

$$\mu_2(w_{\text{remain}}) := \max\{k \in \mathbb{Z}_{\geq 0} \mid \Delta_{1,s}^{\mu_1(w_{\text{remain}})} \cdot \Delta_{2,s}^k \mid w_{\text{remain}}\}.$$

We write

$$w = w_{\text{remain}} = \Delta_{1,s}^{\mu_1(w_{\text{remain}})} \cdot \Delta_{2,s}^{\mu_2(w_{\text{remain}})} \cdot w'_{\text{remain}}.$$

From the Proposition 4.4, we say that the element w'_{remain} belongs to the set $\mathcal{W}_{m,n}$. We consider the following six cases.

Case 1. $L(w) = \{t_1, \dots, t_m, u_\tau\}$. ($u_\tau \in L_2$)

The element w'_{remain} contains the letter s . We remark that the element $\Delta_{2,s}^{\mu_2(w_{\text{remain}})} \cdot w'_{\text{remain}}$ belongs to the set $\mathcal{W}_{m,n}$. The element $\Delta_{2,s}^{\mu_2(w_{\text{remain}})} \cdot w'_{\text{remain}}$ can be written by $w_0(\underline{u}) \cdot s \cdot w_1(\underline{t}) \cdot w_1(\underline{u}) \cdot s \cdots s \cdot w_N(\underline{t}) \cdot w_N(\underline{u})$ ($w_0(\underline{u}) \neq \varepsilon$). We take an element A in $\text{Trans}(w)$. By definition, there exists an element Q in $G_{m,n}^+$ such that an equation $AQ = wA$ holds. Since the element A is not ε , there exists a letter l_0 in L_0 such that the letter l_0 divides A from the left. We write $A = l_0 \cdot A'$. Since $L(w) = \{t_1, \dots, t_m, u_\tau\}$, we have $\{t_1, \dots, t_m, u_\tau\} \subseteq \text{Trans}^{\min}(w)$. We consider the following two cases.

Case 1 – 1. $l_0 = s$.

We consider an equation

$$s \cdot A' \cdot Q = \Delta_{1,s}^{\mu_1(w_{\text{remain}})} \cdot \Delta_{2,s}^{\mu_2(w_{\text{remain}})} \cdot w'_{\text{remain}} \cdot s \cdot A'.$$

Due to the Lemma 4.10, we say that there exists a word Z_1 such that

$$A = s \cdot \Delta_{2, \tilde{C}_2(w_N(\underline{u}) \cdot s)} \cdot Z_1.$$

We say that the element $s \cdot \Delta_{2, \tilde{C}_2(w_N(\underline{u}) \cdot s)}$ belongs to the set $\text{Trans}(w)$. Since it divides Δ from the left, we have verified the property $(P(w; \Delta))$.

Case 1 – 2. $L_0 = u_\sigma$ ($u_\sigma \neq u_\tau$).

We consider an equation

$$u_\sigma \cdot A' \cdot Q = \Delta_{1,s}^{\mu_1(w_{\text{remain}})} \cdot \Delta_{2,s}^{\mu_2(w_{\text{remain}})} \cdot w'_{\text{remain}} \cdot A.$$

Due to the Lemma 4.9, we consider the following two cases.

Case 1 – 2 – 1. $w_N(\underline{u}) \mid \Delta_{2, \tilde{C}_2(w_{N-1}(\underline{u}) \cdot s)}$

We say that there exists a word Z_1 such that

$$A = \Delta_{2, \tilde{C}_2(w_{N-1}(\underline{u}) \cdot s) \cdot w_N(\underline{u})} \cdot Z_1.$$

We say that the element $\Delta_{2, \tilde{C}_2(w_{N-1}(\underline{u}) \cdot s) \cdot w_N(\underline{u})}$ belongs to the set $\text{Trans}(w)$. Since the element $\Delta_{2, \tilde{C}_2(w_{N-1}(\underline{u}) \cdot s) \cdot w_N(\underline{u})}$ divides Δ from the left, we have verified the property $(P(w; \Delta))$.

Case 1 – 2 – 2. $w_N(\underline{u}) \not\vdash_l \Delta_{2, \tilde{C}_2(w_{N-1}(\underline{u}) \cdot s)}$

We say that there exist $w'_0(\underline{t})$ in $F_{1, \text{rm}}^+$ and a word Z_1 such that

$$A \doteq w'_0(\underline{t}) \cdot \Delta_{2, C_2(w_N(\underline{u}))} \cdot Z_1.$$

Since $\{t_1, \dots, t_m, u_\tau\} \subseteq \text{Trans}^{\min}(w)$, we should consider $w'_0(\underline{t}) = \varepsilon$. We say that the element $\Delta_{2, C_2(w_N(\underline{u}))}$ belongs to the set $\text{Trans}(w)$. Since the element $\Delta_{2, C_2(w_N(\underline{u}))}$ divides Δ from the left, we have verified the property $(P(w; \Delta))$.

Case 2. $L(w) = \{u_1, \dots, u_n, t_\tau\}$. ($t_\tau \in L_1$)

We can verify the property $(P(w; \Delta))$ in the same way as the case 1.

Case 3. $L(w) = \{t_\tau, u_\sigma\}$. ($t_\tau \in L_1, u_\sigma \in L_2$)

We remark that the element $\Delta_{1, s}^{\mu_1(w_{\text{remain}})} \cdot \Delta_{2, s}^{\mu_2(w_{\text{remain}})} \cdot w'_{\text{remain}}$ belongs to the set $\mathcal{W}_{m, n}$. It can be written by $w_0(\underline{t}) \cdot w_0(\underline{u}) \cdot s \cdot w_1(\underline{t}) \cdot w_1(\underline{u}) \cdot s \cdots s \cdot w_N(\underline{t}) \cdot w_N(\underline{u})$ ($w_0(\underline{t}) \neq \varepsilon, w_0(\underline{u}) \neq \varepsilon$). We take an element A in $\text{Trans}(w)$. By definition, there exists an element Q in $G_{m, n}^+$ such that an equation $AQ = wA$ holds. Since the element A is not ε , there exists a letter l_0 in L_0 such that the letter l_0 divides A from the left. We write $A = l_0 \cdot A'$. Since $L(w) = \{t_\tau, u_\sigma\}$, we have $\{t_\tau, u_\sigma\} \subseteq \text{Trans}^{\min}(w)$. We consider the following three cases.

Case 3 – 1. $l_0 = s$.

We consider an equation

$$s \cdot A' \cdot Q \doteq w_{\text{remain}} \cdot s \cdot A'.$$

Due to the Lemma 4.10, we say that there exists a word Z_1 such that

$$A \doteq s \cdot \Delta_{1, \tilde{C}_1(w_N(\underline{t}) \cdot s)} \cdot \Delta_{2, \tilde{C}_2(w_N(\underline{u}) \cdot s)} \cdot Z_1.$$

We say that the element $s \cdot \Delta_{1, \tilde{C}_1(w_N(\underline{t}) \cdot s)} \cdot \Delta_{2, \tilde{C}_2(w_N(\underline{u}) \cdot s)}$ belongs to the set $\text{Trans}(w)$. Since it divides Δ from the left, we have verified the property $(P(w; \Delta))$.

Case 3 – 2. $l_0 = t_{\tau'}$ ($t_{\tau'} \neq t_\tau$).

We consider an equation

$$t_{\tau'} \cdot A' \cdot Q \doteq w_{\text{remain}} \cdot A.$$

Due to the Lemma 4.8, we consider the following two cases.

Case 3 – 2 – 1. $w_N(\underline{t}) \vdash_l \Delta_{1, \tilde{C}_1(w_{N-1}(\underline{t}) \cdot s)}$

We say that there exists a word Z_1 such that

$$A \doteq \Delta_{1, \tilde{C}_1(w_{N-1}(\underline{t}) \cdot s) \cdot w_N(\underline{t})} \cdot Z_1.$$

We say that the element $\Delta_{1, \tilde{C}_1(w_{N-1}(\underline{t}) \cdot s) \cdot w_N(\underline{t})}$ belongs to the set $\text{Trans}(w)$. Since it divides Δ from the left, we have verified the property $(P(w; \Delta))$.

Case 3 – 2 – 2. $w_N(\underline{t}) \not\vdash_l \Delta_{1, \tilde{C}_1(w_{N-1}(\underline{t}) \cdot s)}$

We say that there exist $w'_0(\underline{u})$ in $F_{2, \text{rm}}^+$ and a word Z_1 such that

$$A \doteq w'_0(\underline{u}) \cdot \Delta_{1, C_1(w_N(\underline{t}))} \cdot Z_1.$$

Then, we consider an equation

$$w'_0(\underline{u}) \cdot \Delta_{1, C_1(w_N(\underline{t}))} \cdot Z_1 \cdot Q \doteq w_{\text{remain}} \cdot w'_0(\underline{u}) \cdot \Delta_{1, C_1(w_N(\underline{t}))} \cdot Z_1.$$

We have to consider the following two cases.

Case 3 – 2 – 2 – 1. $w'_0(\underline{u}) = \varepsilon$.

Then, we have

$$\Delta_{1, C_1(w_N(\underline{t}))} \cdot Z_1 \cdot Q \doteq w_{\text{remain}} \cdot \Delta_{1, C_1(w_N(\underline{t}))} \cdot Z_1$$

$$\begin{aligned}
&= w_0(\underline{t}) \cdot w_0(\underline{u}) \cdot s \cdot w_1(\underline{t}) \cdot w_1(\underline{u}) \cdot s \cdots s \cdot w_N(\underline{t}) \cdot w_N(\underline{u}) \cdot \Delta_{1, C_1(w_N(\underline{t}))} \cdot Z_1 \\
&= \Delta_{1, s} \cdot w_0(\underline{u}) \cdot s \cdot w_0(\underline{t}) \cdot w_1(\underline{u}) \cdot s \cdots s \cdot w_{N-1}(\underline{t}) \cdot w_N(\underline{u}) \cdot s \cdot R_1(w_N(\underline{t})) \cdot Z_1.
\end{aligned}$$

Since the element $w_0(\underline{u})$ is not ε , due to the Lemma 4.9, we say that there exists a word Z_2 such that

$$Z_1 = \Delta_{2, \tilde{C}_2(w_N(\underline{u}) \cdot s)} \cdot Z_2.$$

We say that the element $\Delta_{1, C_1(w_N(\underline{t}))} \cdot \Delta_{2, \tilde{C}_2(w_N(\underline{u}) \cdot s)}$ belongs to the set $\text{Trans}(w)$. Since it divides Δ from the left, we have verified the property $(P(w; \Delta))$.

Case 3 - 2 - 2 - 2. $w'_0(\underline{u}) \neq \varepsilon$.

Due to the Lemma 4.9, there exists a word Z_2 such that

$$Z_1 = \Delta_{2, \tilde{C}_2(w_N(\underline{u}) \cdot w'_0(\underline{u}) \cdot s)} \cdot Z_2.$$

If $\tilde{C}_2(w_N(\underline{u}) \cdot w'_0(\underline{u}) \cdot s) = \tilde{C}_2(w'_0(\underline{u}) \cdot s)$, then we have

$$\begin{aligned}
&w'_0(\underline{u}) \cdot \Delta_{1, C_1(w_N(\underline{t}))} \cdot \Delta_{2, \tilde{C}_2(w'_0(\underline{u}) \cdot s)} \\
&= \Delta_{1, C_1(w_N(\underline{t}))} \cdot \Delta_{2, s} \cdot \tilde{R}_2(w'_0(\underline{u}) \cdot s).
\end{aligned}$$

We say that the element $\Delta_{1, C_1(w_N(\underline{t}))} \cdot \Delta_{2, s}$ belongs to the set $\text{Trans}(w)$. Since it divides Δ from the left, we have verified the property $(P(w; \Delta))$. If $\tilde{C}_2(w_N(\underline{u}) \cdot w'_0(\underline{u}) \cdot s)$ is not $\tilde{C}_2(w'_0(\underline{u}) \cdot s)$, then we say that the element $w'_0(\underline{u}) \cdot \Delta_{1, C_1(w_N(\underline{t}))} \cdot \Delta_{2, \tilde{C}_2(w_N(\underline{u}) \cdot w'_0(\underline{u}) \cdot s)}$ belongs to the set $\text{Trans}(w)$. Since it divides Δ from the left, we have verified the property $(P(w; \Delta))$.

Case 3 - 3. $l_0 = u_{\sigma'} (u_{\sigma'} \neq u_{\sigma})$.

We can verify the property $(P(w; \Delta))$ in the same way as the case 3 - 2.

Case 4. $L(w) = \{t_\tau\}$. ($t_\tau \in L_1$)

We remark that the element $w = w_{\text{remain}}$ belongs to the set $\mathcal{W}_{m, n}$. It can be written by $w_0(\underline{t}) \cdot s \cdot w_1(\underline{t}) \cdot w_1(\underline{u}) \cdot s \cdots s \cdot w_N(\underline{t}) \cdot w_N(\underline{u})$ ($w_0(\underline{t}) \neq \varepsilon$). We take an element A in $\text{Trans}(w)$. By definition, there exists an element Q in $G_{m, n}^+$ such that an equation $AQ = wA$ holds. Since the element A is not ε , there exists a letter l_0 in L_0 such that the letter l_0 divides A from the left. We write $A = l_0 \cdot A'$. We consider the following three cases.

Case 4 - 1. $l_0 = s$.

Due to the Lemma 4.10, we easily verify the property $(P(w; \Delta))$.

Case 4 - 2. $l_0 = t_{\tau'} (t_{\tau'} \neq t_\tau)$.

We consider an equation

$$t_{\tau'} \cdot A' \cdot Q = w_{\text{remain}} \cdot A.$$

Due to the Lemma 4.8, we consider the following two cases.

Case 4 - 2 - 1. $w_N(\underline{t}) \mid_l \Delta_{1, \tilde{C}_1(w_{N-1}(\underline{t}) \cdot s)}$

We say that there exists a word Z_1 such that

$$A = \Delta_{1, \tilde{C}_1(w_{N-1}(\underline{t}) \cdot s) \cdot w_N(\underline{t})} \cdot Z_1.$$

We say that the element $\Delta_{1, \tilde{C}_1(w_{N-1}(\underline{t}) \cdot s) \cdot w_N(\underline{t})}$ belongs to the set $\text{Trans}(w)$. Since it divides Δ from the left, we have verified the property $(P(w; \Delta))$.

Case 4 - 2 - 2. $w_N(\underline{t}) \nmid_l \Delta_{1, \tilde{C}_1(w_{N-1}(\underline{t}) \cdot s)}$

We say that there exist $w'_0(\underline{u})$ in $F_{2, \text{rm}}^+$ and a word Z_1 such that

$$A = w'_0(\underline{u}) \cdot \Delta_{1, C_1(w_N(\underline{t}))} \cdot Z_1.$$

If $w'_0(\underline{u})$ is ε , due to the assumption that $w_0(\underline{u})$ is ε , then we easily show that the element $\Delta_{1,C_1(w_N(\underline{t}))}$ belongs to the set $\text{Trans}(w)$. Otherwise, we consider an equation

$$w'_0(\underline{u}) \cdot \Delta_{1,C_1(w_N(\underline{t}))} \cdot Z_1 \cdot Q = w_{\text{remain}} \cdot w'_0(\underline{u}) \cdot \Delta_{1,C_1(w_N(\underline{t}))} \cdot Z_1.$$

Due to the Lemma 4.9, there exists a word Z_2 such that

$$Z_1 = \Delta_{2,\tilde{C}_2(w_N(\underline{u}) \cdot w'_0(\underline{u}) \cdot s)} \cdot Z_2.$$

We have to consider the following two cases.

Case 4 - 2 - 2 - 1. $\tilde{C}_2(w_N(\underline{u}) \cdot w'_0(\underline{u}) \cdot s) = \tilde{C}_2(w'_0(\underline{u}) \cdot s)$

Then, we have

$$\begin{aligned} & w'_0(\underline{u}) \cdot \Delta_{1,C_1(w_N(\underline{t}))} \cdot \Delta_{2,\tilde{C}_2(w'_0(\underline{u}) \cdot s)} \\ &= \Delta_{1,C_1(w_N(\underline{t}))} \cdot \Delta_{2,s} \cdot \tilde{R}_2(w'_0(\underline{u}) \cdot s). \end{aligned}$$

We say that the element $\Delta_{1,C_1(w_N(\underline{t}))} \cdot \Delta_{2,s}$ belongs to the set $\text{Trans}(w)$. Since it divides Δ from the left, we have verified the property $(P(w; \Delta))$.

Case 4 - 2 - 2 - 2. $\tilde{C}_2(w_N(\underline{u}) \cdot w'_0(\underline{u}) \cdot s)$ is not $\tilde{C}_2(w'_0(\underline{u}) \cdot s)$

Then, we say that the element $w'_0(\underline{u}) \cdot \Delta_{1,C_1(w_N(\underline{t}))} \cdot \Delta_{2,\tilde{C}_2(w_N(\underline{u}) \cdot w'_0(\underline{u}) \cdot s)}$ belongs to the set $\text{Trans}(w)$. Since it divides Δ from the left, we have verified the property $(P(w; \Delta))$.

Case 4 - 3. $l_0 = u_{\sigma'} \ (u_{\sigma'} \neq u_{\sigma})$.

We consider an equation

$$u_{\sigma'} \cdot A' \cdot Q = w_{\text{remain}} \cdot A.$$

Due to the Lemma 4.9, we consider the following two cases.

Case 4 - 3 - 1. $w_N(\underline{u}) \mid_l \Delta_{2,\tilde{C}_2(w_{N-1}(\underline{u}) \cdot s)}$

We say that there exists a word Z_1 such that

$$A = \Delta_{2,\tilde{C}_2(w_{N-1}(\underline{u}) \cdot s) \cdot w_N(\underline{u})} \cdot Z_1.$$

We say that the element $\Delta_{2,\tilde{C}_2(w_{N-1}(\underline{u}) \cdot s) \cdot w_N(\underline{u})}$ belongs to the set $\text{Trans}(w)$. Since it divides Δ from the left, we have verified the property $(P(w; \Delta))$.

Case 4 - 3 - 2. $w_N(\underline{u}) \not\mid_l \Delta_{2,\tilde{C}_2(w_{N-1}(\underline{u}) \cdot s)}$

We say that there exist $w'_0(\underline{t})$ in $F_{1,\text{rm}}^+$ and a word Z_1 such that

$$A = w'_0(\underline{t}) \cdot \Delta_{2,C_2(w_N(\underline{u}))} \cdot Z_1.$$

Then, we consider an equation

$$w'_0(\underline{t}) \cdot \Delta_{2,C_2(w_N(\underline{u}))} \cdot Z_1 \cdot Q = w_{\text{remain}} \cdot w'_0(\underline{t}) \cdot \Delta_{2,C_2(w_N(\underline{u}))} \cdot Z_1.$$

We have to consider the following two cases.

Case 4 - 3 - 2 - 1. $w'_0(\underline{t}) = \varepsilon$.

Then, we have

$$\begin{aligned} & \Delta_{2,C_2(w_N(\underline{u}))} \cdot Z_1 \cdot Q = w_{\text{remain}} \cdot \Delta_{2,C_2(w_N(\underline{u}))} \cdot Z_1 \\ &= w_0(\underline{t}) \cdot w_0(\underline{u}) \cdot s \cdot w_1(\underline{t}) \cdot w_1(\underline{u}) \cdot s \cdots s \cdot w_N(\underline{t}) \cdot w_N(\underline{u}) \cdot \Delta_{2,C_2(w_N(\underline{u}))} \cdot Z_1 \\ &= \Delta_{2,s} \cdot w_0(\underline{t}) \cdot s \cdot w_0(\underline{u}) \cdot w_1(\underline{t}) \cdot s \cdots s \cdot w_{N-1}(\underline{u}) \cdot w_N(\underline{t}) \cdot s \cdot R_2(w_N(\underline{u})) \cdot Z_1. \end{aligned}$$

Since the element $w_0(\underline{t})$ is not ε , due to the Lemma 4.8, we say that there exists a word Z_2 such that

$$Z_1 = \Delta_{1,\tilde{C}_1(w_N(\underline{t}) \cdot s)} \cdot Z_2.$$

We say that the element $\Delta_{2,C_2(w_N(\underline{u}))} \cdot \Delta_{1,\tilde{C}_1(w_N(\underline{t}) \cdot s)}$ belongs to the set $\text{Trans}(w)$. Since it divides Δ from the left, we have verified the property $(P(w; \Delta))$.

Case 4 - 3 - 2 - 2. $w'_0(\underline{t})$ is not ε .

Due to the Lemma 4.8, there exists a word Z_2 such that

$$Z_1 = \Delta_{1,\tilde{C}_1(w_N(\underline{t}) \cdot w'_0(\underline{t}) \cdot s)} \cdot Z_2.$$

If $\tilde{C}_1(w_N(\underline{t}) \cdot w'_0(\underline{t}) \cdot s) = \tilde{C}_1(w'_0(\underline{t}) \cdot s)$, then we have

$$\begin{aligned} & w'_0(\underline{t}) \cdot \Delta_{2,C_2(w_N(\underline{u}))} \cdot \Delta_{1,\tilde{C}_1(w'_0(\underline{t}) \cdot s)} \\ &= \Delta_{2,C_2(w_N(\underline{u}))} \cdot \Delta_{1,s} \cdot \tilde{R}_1(w'_0(\underline{t}) \cdot s). \end{aligned}$$

We easily show that the element $\Delta_{2,C_2(w_N(\underline{u}))} \cdot \Delta_{1,s}$ belongs to the set $\text{Trans}(w)$. Since it divides Δ from the left, we have verified the property $(P(w; \Delta))$. If the element $\tilde{C}_1(w_N(\underline{t}) \cdot w'_0(\underline{t}) \cdot s)$ is not equal to $\tilde{C}_1(w'_0(\underline{t}) \cdot s)$, then we say that the element $w'_0(\underline{t}) \cdot \Delta_{2,C_2(w_N(\underline{u}))} \cdot \Delta_{1,\tilde{C}_1(w_N(\underline{t}) \cdot w'_0(\underline{t}) \cdot s)}$ belongs to the set $\text{Trans}(w)$. Since it divides Δ from the left, we have verified the property $(P(w; \Delta))$.

Case 5. $L(w) = \{u_\sigma\}$ ($u_\sigma \in L_2$).

We can verify the property $(P(w; \Delta))$ in the same way as the case 4.

Case 6. $L(w) = \{s\}$.

We remark that the element $w = w_{\text{remain}}$ belongs to the set $\mathcal{W}_{m,n}$. It can be written by $s \cdot w_1(\underline{t}) \cdot w_1(\underline{u}) \cdot s \cdots s \cdot w_N(\underline{t}) \cdot w_N(\underline{u})$. We take an element A in $\text{Trans}(w)$. By definition, there exists an element Q in $G_{m,n}^+$ such that an equation $AQ = wA$ holds. Since the element A is not ε , there exists a letter l_0 in L_0 such that the letter l_0 divides A from the left. We write $A = l_0 \cdot A'$. We have to consider the following two cases.

Case 6 - 1. $l_0 = \{t_\tau\}$ ($t_\tau \in L_1$).

We consider an equation

$$t_\tau \cdot A' \cdot Q = w_{\text{remain}} \cdot A.$$

Due to the Lemma 4.8, we consider the following two cases.

Case 6 - 1 - 1. $w_N(\underline{t}) \mid_l \Delta_{1,\tilde{C}_1(w_{N-1}(\underline{t}) \cdot s)}$

We say that there exists a word Z_1 such that

$$A = \Delta_{1,\tilde{C}_1(w_{N-1}(\underline{t}) \cdot s) \cdot w_N(\underline{t})} \cdot Z_1.$$

We say that the element $\Delta_{1,\tilde{C}_1(w_{N-1}(\underline{t}) \cdot s) \cdot w_N(\underline{t})}$ belongs to the set $\text{Trans}(w)$. Since it divides Δ from the left, we have verified the property $(P(w; \Delta))$.

Case 6 - 1 - 2. $w_N(\underline{t}) \not\mid_l \Delta_{1,\tilde{C}_1(w_{N-1}(\underline{t}) \cdot s)}$

We say that there exist $w'_0(\underline{u})$ in $F_{2,\text{rm}}^+$ and a word Z_1 such that

$$A = w'_0(\underline{u}) \cdot \Delta_{1,C_1(w_N(\underline{t}))} \cdot Z_1.$$

Then, we consider an equation

$$w'_0(\underline{u}) \cdot \Delta_{1,C_1(w_N(\underline{t}))} \cdot Z_1 \cdot Q = w_{\text{remain}} \cdot w'_0(\underline{u}) \cdot \Delta_{1,C_1(w_N(\underline{t}))} \cdot Z_1.$$

If the element $w'_0(\underline{u})$ is ε , due to the assumption that $w_0(\underline{u})$ is ε , then we easily show that the element $\Delta_{1,C_1(w_N(\underline{t}))}$ belongs to the set $\text{Trans}(w)$. Otherwise, we consider an equation

$$w'_0(\underline{u}) \cdot \Delta_{1,C_1(w_N(\underline{t}))} \cdot Z_1 \cdot Q = w_{\text{remain}} \cdot w'_0(\underline{u}) \cdot \Delta_{1,C_1(w_N(\underline{t}))} \cdot Z_1.$$

Due to the Lemma 4.9, there exists a word Z_2 such that

$$Z_1 = \Delta_{2, \tilde{C}_2(w_N(\underline{u}) \cdot w'_0(\underline{u}) \cdot s)} \cdot Z_2.$$

Case 6 - 1 - 2 - 1. $\tilde{C}_2(w_N(\underline{u}) \cdot w'_0(\underline{u}) \cdot s) = \tilde{C}_2(w'_0(\underline{u}) \cdot s)$

Then, we have

$$\begin{aligned} & w'_0(\underline{u}) \cdot \Delta_{1, C_1(w_N(\underline{t}))} \cdot \Delta_{2, \tilde{C}_2(w'_0(\underline{u}) \cdot s)} \\ &= \Delta_{1, C_1(w_N(\underline{t}))} \cdot \Delta_{2, s} \cdot \tilde{R}_2(w'_0(\underline{u}) \cdot s). \end{aligned}$$

We say that the element $\Delta_{1, C_1(w_N(\underline{t}))} \cdot \Delta_{2, s}$ belongs to the set $\text{Trans}(w)$. Since it divides Δ from the left, we have verified the property $(P(w; \Delta))$.

Case 6 - 1 - 2 - 2. $\tilde{C}_2(w_N(\underline{u}) \cdot w'_0(\underline{u}) \cdot s)$ is not $\tilde{C}_2(w'_0(\underline{u}) \cdot s)$

Then, we say that the element $w'_0(\underline{u}) \cdot \Delta_{1, C_1(w_N(\underline{t}))} \cdot \Delta_{2, \tilde{C}_2(w_N(\underline{u}) \cdot w'_0(\underline{u}) \cdot s)}$ belongs to the set $\text{Trans}(w)$. Since it divides Δ from the left, we have verified the property $(P(w; \Delta))$.

Case 6 - 2. $l_0 = \{u_\sigma\} \ (u_\sigma \in L_2)$.

We can verify the property $(P(w; \Delta))$ in the same way as the case 6 - 1.

These complete the proof. By verifying the property $(P(w; \Delta))$ for an arbitrary element w in the monoid $G_{m,n}^+$, we show that the conjugacy problem in it is solvable. Thus, we conclude that the conjugacy problem in the group $G_{m,n}$ is solvable.

Question 4.11. In [12], the author studied a positive homogeneously presented cancellative monoid H_n^+ that carries a unique minimal fundamental element Δ . Thanks to the reduction lemma for it, we easily show that the monoid H_n^+ does not satisfy the LCM condition and $\mathcal{F}(H_n^+) = \mathcal{QZ}(H_n^+) \setminus \{\varepsilon\}$. Hence, the idealistic subsemigroup $\mathcal{F}(H_n^+)(\subseteq \mathcal{QZ}(H_n^+))$ is singly generated by Δ and the monoid H_n^+ is tame. Is the property $(P(w; \Delta))$ satisfied for an arbitrary element w in the monoid?

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